

# Global in Time Existence of Self-Interacting Scalar Field in De Sitter Spacetimes

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## Abstract

We prove the existence of a global in time solution of the semilinear Klein-Gordon equation in the de Sitter space-time. The coefficients of the equation depend on spatial variables as well, that make results applicable to the space-time with the time slices being Riemannian manifolds.

## 0 Introduction

In this paper we prove the existence of a global in time solution of the semilinear Klein-Gordon equation in the de Sitter space-time. The coefficients of the equation depend on spatial variables as well, that make results applicable to the space-time with the time slices being Riemannian manifolds. In the spatially flat de Sitter model, these slices are  $\mathbb{R}^3$ , while in the spatially closed and spatially open cases these slices can be the three-sphere  $\mathbb{S}^3$  and the three-hyperboloid  $\mathbb{H}^3$ , respectively (see, e.g., [12, p.113]).

The metric  $g$  in the de Sitter space-time is defined as follows,  $g_{00} = g^{00} = -1$ ,  $g_{0j} = g^{0j} = 0$ ,  $g_{ij}(x, t) = e^{2t}\sigma_{ij}(x)$ ,  $i, j = 1, 2, \dots, n$ , where  $\sum_{j=1}^n \sigma^{ij}(x)\sigma_{jk}(x) = \delta_{ik}$ , and  $\delta_{ij}$  is Kronecker's delta. The metric  $\sigma^{ij}(x)$  describes the time slices. In the quantum field theory the matter fields are described by a function  $\psi$  that must satisfy equations of motion. In the case of a massive scalar field, the equation of motion is the semilinear Klein-Gordon equation generated by the metric  $g$ :

$$\square_g \psi = m^2 \psi + V'_\psi(x, \psi).$$

Here  $m$  is a physical mass of the particle. In physical terms this equation describes a local self-interaction for a scalar particle. A typical example of a potential function would be  $V(\psi) = \psi^4$ . The semilinear equations are also commonly used models for general nonlinear problems.

The covariant Klein-Gordon equation in the de Sitter space-time in the coordinates is

$$\psi_{tt} - \frac{e^{-2t}}{\sqrt{|\det \sigma(x)|}} \sum_{i,j=1}^n \frac{\partial}{\partial x^i} \left( \sqrt{|\det \sigma(x)|} \sigma^{ij}(x) \frac{\partial}{\partial x^j} \psi \right) + n\psi_t + m^2 \psi = F(\psi).$$

This is a special case of the equation

$$\psi_{tt} + n\psi_t - e^{-2t}A(x, \partial_x)\psi + m^2\psi = F(\psi),$$

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where  $A(x, \partial_x) = \sum_{|\alpha| \leq 2} a_\alpha(x) \partial_x^\alpha$  is a second order partial differential operator. More precisely, in this paper we assume that  $a_\alpha(x)$ ,  $|\alpha| = 2$ , is positive definite (and symmetric).

In [32]-[34] a global existence of small data solutions of the Cauchy problem for the semilinear Klein-Gordon equation and systems of equations in the de Sitter space-time with flat time slices, that is,  $\sigma^{ij}(x) = \delta^{ij}$ , is proved. The nonlinearity  $F$  was assumed Lipschitz continuous with exponent  $\alpha \geq 0$  (see definition below). It was discovered that unlike the same problem in the Minkowski space-time, no restriction on the order of nonlinearity is required, provided that a physical mass of the field belongs to some set,  $m \in (0, \sqrt{n^2 - 1}/2] \cup [n/2, \infty)$ . For  $n = 3$  the mass  $m$  interval  $(0, \sqrt{2})$  is called the Higuchi bound in quantum field theory [18]. The proof of the global existence in [32]-[34] is based on the special integral representations (see Section 1) and  $L^p - L^q$  estimates.

In the present paper we are going to extend the small data global existence result of [34] for the spatially flat de Sitter space-time to the de Sitter space-time with the time slices being Riemannian manifolds.

To formulate the main theorem of this paper we need the following description of the nonlinear term. Let  $B_p^{s,q}$  be the Besov space.

**Condition ( $\mathcal{L}$ ).** *The function  $F$  is said to be Lipschitz continuous with exponent  $\alpha \geq 0$  in the space  $B_p^{s,q}$  if there is a constant  $C \geq 0$  such that*

$$\|F(x, \psi_1(x)) - F(x, \psi_2(x))\|_{B_p^{s,q}} \leq C \|\psi_1 - \psi_2\|_{B_{p'}^{s,q}} \left( \|\psi_1\|_{B_{p'}^{s,q}}^\alpha + \|\psi_2\|_{B_{p'}^{s,q}}^\alpha \right) \quad (0.1)$$

for all  $\psi_1, \psi_2 \in B_{p'}^{s,q}$ , where  $1/p + 1/p' = 1$ .

For the case of  $B_2^{s,2} = H_{(s)}(\mathbb{R}^n)$ , define the complete metric space

$$X(R, s, \gamma) := \{ \psi \in C([0, \infty); H_{(s)}(\mathbb{R}^n)) \mid \|\psi\|_X := \sup_{t \in [0, \infty)} e^{\gamma t} \|\psi(x, t)\|_{H_{(s)}(\mathbb{R}^n)} \leq R \},$$

$\gamma \geq 0$ , with the metric

$$d(\psi_1, \psi_2) := \sup_{t \in [0, \infty)} e^{\gamma t} \|\psi_1(x, t) - \psi_2(x, t)\|_{H_{(s)}(\mathbb{R}^n)}.$$

We denote  $\mathcal{B}^\infty$  the space of all  $C^\infty(\mathbb{R}^n)$  functions with uniformly bounded derivatives of all orders.

**Theorem 0.1** *Let  $A(x, \partial_x) = \sum_{|\alpha| \leq 2} a_\alpha(x) \partial_x^\alpha$  be a second order negative elliptic differential operator with real coefficients  $a_\alpha \in \mathcal{B}^\infty$ . Assume that the nonlinear term  $F(u)$  is Lipschitz continuous with exponent  $\alpha > 0$  in the space  $H_{(s)}(\mathbb{R}^n)$ ,  $s > n/2 \geq 1$ , and  $F(0) = 0$ . Assume also that  $m \in (0, \sqrt{n^2 - 1}/2] \cup [n/2, \infty)$ . Then there exists  $\varepsilon_0 > 0$  such that, for every given functions  $\psi_0, \psi_1 \in H_{(s)}(\mathbb{R}^n)$ , such that*

$$\|\psi_0\|_{H_{(s)}(\mathbb{R}^n)} + \|\psi_1\|_{H_{(s)}(\mathbb{R}^n)} \leq \varepsilon, \quad \varepsilon < \varepsilon_0,$$

*there exists a global solution  $\psi \in C^1([0, \infty); H_{(s)}(\mathbb{R}^n))$  of the Cauchy problem*

$$\psi_{tt} + n\psi_t - e^{-2t} A(x, \partial_x) \psi + m^2 \psi = F(x, \psi), \quad (0.2)$$

$$\psi(x, 0) = \psi_0(x), \quad \psi_t(x, 0) = \psi_1(x). \quad (0.3)$$

*The solution  $\psi(x, t)$  belongs to the space  $X(2\varepsilon, s, \gamma)$ , that is,*

$$\sup_{t \in [0, \infty)} e^{\gamma t} \|\psi(\cdot, t)\|_{H_{(s)}(\mathbb{R}^n)} < 2\varepsilon,$$

with  $\gamma$  such that either  $0 < \gamma \leq \frac{1}{\alpha+1} \left( \frac{n}{2} - \sqrt{\frac{n^2}{4} - m^2} \right)$  if  $\sqrt{n^2-1}/2 \geq m > 0$ , or we choose  $0 \leq \gamma_0 < \frac{n-1}{2}$  if  $m = n/2$  and  $0 \leq \gamma_0 \leq \frac{n-1}{2}$  if  $m > n/2$ , then  $\gamma \leq \min \left\{ \gamma_0, \frac{n}{2(\alpha+1)} \right\}$ .

If  $m \in (\sqrt{n^2-1}/2, n/2)$ , then for the problem with  $\psi_0 = 0$  the global solution exists and belongs to  $X(2\varepsilon, s, \gamma)$ , where  $\gamma \in (0, \frac{1}{\alpha+1}(\frac{n}{2} - \sqrt{\frac{n^2}{4} - m^2}))$ .

The range  $m \in (\sqrt{n^2-1}/2, n/2)$ , which seems to be a forbidden mass interval for the problem with general initial data, can be allowed if we change the setting of the problem. Indeed, if we consider the initial value problem with vanishing Cauchy data and with the source term  $f$ , then we have the following result for all  $m > 0$ .

**Theorem 0.2** Let  $A(x, \partial_x) = \sum_{|\alpha| \leq 2} a_\alpha(x) \partial_x^\alpha$  be a second order negative elliptic differential operator with real coefficients  $a_\alpha \in \mathcal{B}^\infty$ . Assume that the nonlinear term  $F(u)$  is a Lipschitz continuous with exponent  $\alpha > 0$  in the space  $H_{(s)}(\mathbb{R}^n)$ ,  $s > n/2 \geq 1$ , and  $F(0) = 0$ . Assume also that  $m > 0$ . Then there exists  $\varepsilon_0 > 0$  such that, for every given function  $f \in X(\varepsilon, s, \gamma_{rhs})$ , such that

$$\sup_{t \in [0, \infty)} e^{\gamma_{rhs} t} \|f(x, t)\|_{H_{(s)}(\mathbb{R}^n)} \leq \varepsilon < \varepsilon_0,$$

there exists a global solution  $\psi \in C^1([0, \infty); H_{(s)}(\mathbb{R}^n))$  of the Cauchy problem

$$\psi_{tt} + n\psi_t - e^{-2t} A(x, \partial_x) \psi + m^2 \psi = f + F(x, \psi), \quad (0.4)$$

$$\psi(x, 0) = 0, \quad \psi_t(x, 0) = 0. \quad (0.5)$$

The solution  $\psi(x, t)$  belongs to the space  $X(2\varepsilon, s, \gamma)$ , that is,

$$\sup_{t \in [0, \infty)} e^{\gamma t} \|\psi(\cdot, t)\|_{H_{(s)}(\mathbb{R}^n)} < 2\varepsilon,$$

with  $\gamma$  such that

$$\left\{ \begin{array}{ll} \gamma < \frac{1}{\alpha+1} \gamma_{rhs} & \text{if } m < \frac{n}{2} \text{ and } \gamma_{rhs} \leq \frac{n}{2} - \sqrt{\frac{n^2}{4} - m^2}, \\ \gamma < \frac{1}{\alpha+1} \left( \frac{n}{2} - \sqrt{\frac{n^2}{4} - m^2} \right) & \text{if } m < \frac{n}{2} \text{ and } \gamma_{rhs} > \frac{n}{2} - \sqrt{\frac{n^2}{4} - m^2}, \\ \gamma \leq \min \left\{ \gamma_{rhs}, \frac{n}{2(\alpha+1)} \right\} & \text{if } m \geq \frac{n}{2} \text{ and } \frac{n}{2} > \gamma_{rhs}, \\ \gamma \leq \min \left\{ \gamma_0, \frac{n}{2(\alpha+1)} \right\} & \text{where } \gamma_0 < \gamma_{rhs} \text{ if } m = \frac{n}{2} \text{ and } \frac{n}{2} = \gamma_{rhs}, \\ \gamma \leq \frac{n}{2(\alpha+1)} & \text{if } m > \frac{n}{2} \text{ and } \frac{n}{2} < \gamma_{rhs}, \\ \gamma < \frac{n}{2(\alpha+1)} & \text{if } m = \frac{n}{2} \text{ and } \frac{n}{2} < \gamma_{rhs}, \\ \gamma \leq \frac{n}{2(\alpha+1)} & \text{if } m > \frac{n}{2} \text{ and } \frac{n}{2} = \gamma_{rhs}. \end{array} \right.$$

The main tools to prove Theorems 0.1, 0.2 are the following: 1) the integral transform, which produces representations of the solutions of the linear equation, 2) the decay estimates in the Besov spaces, which generate weighted Strichartz estimates, and 3) the fixed point theorem.

The kernel of the integral transform that will be used in this paper (see Section 1) is the following function

$$E(x, t; x_0, t_0; M) := 4^{-M} e^{M(t_0+t)} \left( (e^{-t_0} + e^{-t})^2 - (x - x_0)^2 \right)^{M-\frac{1}{2}} \\ \times F\left(\frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(e^{-t_0} - e^{-t})^2 - (x - x_0)^2}{(e^{-t_0} + e^{-t})^2 - (x - x_0)^2}\right),$$

where  $(x, t) \in D_+(x_0, t_0) \cup D_-(x_0, t_0)$ ,  $D_+(x_0, t_0)$  and  $D_-(x_0, t_0)$  are the *chronological future* and the *chronological past*, respectively, of the point  $(x_0, t_0) \in \mathbb{R}^{n+1}$ ,  $M \in \mathbb{C}$ , and  $F(a, b; c; \zeta)$  is the hypergeometric function. The values of the physical mass  $m$ , which lead to the values of  $M = -k + \frac{1}{2}$ ,  $k = 0, 1, 2, \dots$ , are called in [31] *the knot points*. One of these knot points,  $m = \sqrt{n^2 - 1}/2$ , presents the only field that obeys the Huygens' principle [31]. For these values of the curved mass  $M$  the functions  $F(-k, -k; 1; z)$ ,  $k = 0, 1, 2, \dots$ , are polynomials.

It is known that the Klein-Gordon quantum fields whose squared physical masses are negative (imaginary mass) represent tachyons. (See, e.g., [8].) In [8] the Klein-Gordon equation with imaginary mass is considered, and it is shown that the localized disturbances spread with at most the speed of light, but grow exponentially. The conclusion is made that free tachyons have to be rejected on stability grounds.

The Klein-Gordon quantum fields on the de Sitter manifold with imaginary mass, which take an infinite set of discrete values as follows

$$m^2 = -k(k + n), \quad k = 0, 1, 2, \dots, \quad (0.6)$$

present a family of scalar tachyonic quantum fields. Epstein and Moschella [14] give a complete study of a family of scalar tachyonic quantum fields which are linear Klein-Gordon quantum fields on the de Sitter manifold whose squared masses are negative and take an infinite set of discrete values (0.6). The corresponding linear equation is

$$\psi_{tt} + n\psi_t - e^{-2t}\Delta\psi + m^2\psi = 0,$$

for which the kernel is  $E(x, t; x_0, t_0; M)$ , where  $M = \sqrt{\frac{n^2}{4} + k(k + n)} = k + \frac{n}{2}$ ,  $k = 0, 1, 2, \dots$ . If  $n$  is an odd number, then  $m$  takes value at knot points set. The nonexistence of a global in time solution of the semilinear Klein-Gordon massive tachyonic (quantum fields) equation in the de Sitter space-time is proved in [30]. The conclusion is that the self-interacting tachyons in the de Sitter space-time have finite lifespan. More precisely, consider the semilinear equation

$$\psi_{tt} + n\psi_t - e^{-2t}\Delta\psi - m^2\psi = c|\psi|^{1+\alpha},$$

which is commonly used model for general nonlinear problems. Then, according to Theorem 1.1 [30], if  $c \neq 0$ ,  $\alpha > 0$ , and  $m \neq 0$ , then for every positive numbers  $\varepsilon$  and  $s$  there exist functions  $\psi_0, \psi_1 \in C_0^\infty(\mathbb{R}^n)$  such that  $\|\psi_0\|_{H_{(s)}(\mathbb{R}^n)} + \|\psi_1\|_{H_{(s)}(\mathbb{R}^n)} \leq \varepsilon$  but the solution  $\psi = \psi(x, t)$  with the initial values (0.3) blows up in finite time.

The equation, which is considered in Theorem 0.1 and Theorem 0.2, is more general than the covariant Klein-Gordon equation. Furthermore, these theorems, after evident modification, can be applied to the smooth pseudo Riemannian manifold  $(\mathcal{V}, g)$  of dimension  $n+1$  and  $\mathcal{V} = \mathbb{R} \times \mathcal{S}$  with  $\mathcal{S}$  an  $n$ -dimensional orientable smooth manifold and  $g$  is the de Sitter metric. One important example of the equation on the smooth pseudo Riemannian manifold that is amenable to Theorem 0.1 is if

$\mathcal{S}$  is a non-Euclidean space of constant negative curvature and then the equation of the problem (0.2)&(0.3) is a non-Euclidean Klein-Gordon equation.

Although recently the equations in the de Sitter and anti-de Sitter space-times became the focus of interest for an increasing number of authors (see, e.g., [1, 4, 6, 11, 14, 15, 19, 23, 24, 28] and the bibliography therein) which investigate those equations from a wide spectrum of perspectives, there are very few papers on the semilinear Klein-Gordon equation in the de Sitter space-time. Here we mention some of them closely related to our main result. Baskin [6] discussed small data global energy class solutions for the scalar Klein-Gordon equation on asymptotically de Sitter spaces, which are compact manifolds with boundary. More precisely, in [6] the following Cauchy problem is considered for the semilinear equation

$$\square_g u + m^2 u = f(u), \quad u(x, t_0) = \varphi_0(x) \in H_{(1)}(\mathbb{R}^n), \quad u_t(x, t_0) = \varphi_1(x) \in L^2(\mathbb{R}^n),$$

where mass is large,  $m^2 > n^2/4$ ,  $f$  is a smooth function and satisfies conditions  $|f(u)| \leq c|u|^{\alpha+1}$ ,  $|u| \cdot |f'(u)| \sim |f(u)|$ ,  $f(u) - f'(u) \cdot u \leq 0$ ,  $\int_0^u f(v)dv \geq 0$ , and  $\int_0^u f(v)dv \sim |u|^{\alpha+2}$  for large  $|u|$ . It is also assumed that  $\alpha = \frac{4}{n-1}$ . In Theorem 1.3 [6] the existence of the global solution for small energy data is stated. (For more references on the asymptotically de Sitter spaces, see the bibliography in [5], [28].)

Hintz and Vasy [19] considered the semilinear wave equations of the form

$$(\square_g - \lambda)u = f + q(u, du)$$

on a manifold  $M$ , where  $q$  is a polynomial vanishing at least quadratically at  $(0, 0)$ , in *asymptotically* de Sitter and Kerr-de Sitter spaces, as well as asymptotically Minkowski spaces. The initial data for the equation are generated by the source term  $f$ . The linear framework in [19] is based on the b-analysis, in the sense of Melrose, introduced in this context by Vasy to describe the asymptotic behavior of solutions of linear equations. Hintz and Vasy have shown the small source term  $f$  solvability of suitable semilinear wave and Klein-Gordon equations. However, the microlocal, high regularity approach that was taken in [19] does not apply to low regularity non-linearities covered in Theorem 0.1. Their result for *asymptotically* de Sitter space-time and polynomial semilinear term with large  $\alpha$  covers also the range  $m \in (\sqrt{n^2 - 1}/2, n/2)$ . On the other hand, the important case of  $n = 3$  and the quadratic nonlinearity is not covered. We note here that their results as well as Theorem 0.2 of the present paper, neither prove nor disprove the following conjecture from the article [34].

**Conjecture 0.3** [34] *The interval  $(\sqrt{n^2 - 1}/2, n/2)$  is a forbidden mass interval for the small data global solvability of the Cauchy problem for all  $\alpha \in (0, \infty)$  satisfying condition  $(\mathcal{L})$ .*

The case of the massless field  $m = 0$ , that is, the global existence for the semilinear wave equation (0.2) in the de Sitter space-time, is still open. On the other hand, de Sitter space-time has different realizations (see, e.g., [16]), that allow one to reduce the problem to the case of the manifold with constant curvature. For geometric reasons, one can expect better dispersive properties from a linear equation on the manifold with negative constant curvature, and, consequently, stronger results for a semilinear equation, than in the Euclidean setting.

The global existence of the solutions of the equation on the manifold with the time slices being real hyperbolic spaces  $\mathbb{H}^n$  is investigated by Metcalfe and Taylor in [21, 22], and by Anker, Pierfelice, and Vallarino in [2, 3]. In particular, in [21] for the equation  $(\partial_t^2 - \Delta)u = au^2$  on  $\mathbb{R} \times \mathbb{H}^3$  existence of the small data global solution is proved. Thus, it is shown for a range of powers that is broader

than that known for Euclidean space from the so-called Strauss conjecture. Note that the operators in those articles [21, 22, 2, 3] have time-independent coefficients.

The Cauchy problem for the damped linear wave equations with a time-dependent propagation speed and dissipations,  $u_{tt} - a(t)^2 \Delta u + b(t)u_t = 0$ , where  $a \in L^1(0, \infty)$ , is considered by Ebert and Reissig [13]. The analysis of results of [13] hopefully can lead to the global existence in the problem for the wave equation in the de Sitter space-time and can shed a light on the interval  $(\sqrt{n^2 - 1}/2, n/2)$ .

Nakamura [24] considered the Cauchy problem for the semi-linear Klein-Gordon equations in de Sitter space-time with  $n \leq 4$  and with flat time slices. The nonlinear term is of power type for  $n = 3, 4$ , or of exponential type for  $n = 1, 2$ . For the power type semilinear term with  $\frac{4}{n} \leq \alpha \leq \frac{2}{n-2}$  Nakamura [24] proved the existence of global solutions in the energy class.

Galstian and Yagdjian [15] proved the existence of global solutions in the energy class in the case of  $n = 3, 4$  and the nonlinear term is of power type. They considered the equation in the Friedmann-Lemaître-Robertson-Walker space-times (FLRW space-times) with the time slices being Riemannian manifolds. The Klein-Gordon equation in the Einstein-de Sitter and de Sitter space-times are important particular cases discussed in [15].

The present paper is organized as follows. In Section 1 we describe the integral transform and, generated by that transform, representations (from [33]) for the solutions of the Cauchy problem for the linear equation. Then, in Sections 2-4 we derive the  $B_p^{s,q} - B_{p'}^{s',q}$  estimates for the solutions of that equation with large, small and critical (for the Huygens' principle) mass, respectively. These representations are used in the Subsections 4.2 -4.3 for the derivation of asymptotic expansions. The last section, Section 5, is devoted to the solvability of the associated integral equation and to the proof of Theorem 0.1 and Theorem 0.2.

## 1 Integral Transform

If we introduce the new unknown function  $u = e^{nt/2}\psi$ , then the equation (0.2) takes the form of the Klein-Gordon equation

$$u_{tt} - e^{-2t}A(x, \partial_x)u + M^2u = e^{nt/2}F(x, e^{-nt/2}u), \quad (1.1)$$

where  $M^2 = m^2 - \frac{n^2}{4}$  is the square of the so-called curved (or effective) mass.

We introduce the following notations. First, we define a *chronological future*  $D_+(x_0, t_0)$  and a *chronological past*  $D_-(x_0, t_0)$  of the point  $(x_0, t_0)$ ,  $x_0 \in \mathbb{R}^n$ ,  $t_0 \in \mathbb{R}$ , as follows:  $D_\pm(x_0, t_0) := \{(x, t) \in \mathbb{R}^{n+1} ; |x - x_0| \leq \pm(e^{-t_0} - e^{-t})\}$ . Then, for  $(x_0, t_0) \in \mathbb{R}^n \times \mathbb{R}$ ,  $M \in \mathbb{C}$ , we define the function

$$\begin{aligned} E(x, t; x_0, t_0; M) &:= 4^{-M} e^{M(t_0+t)} \left( (e^{-t_0} + e^{-t})^2 - (x - x_0)^2 \right)^{M-\frac{1}{2}} \\ &\times F\left(\frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(e^{-t_0} - e^{-t})^2 - (x - x_0)^2}{(e^{-t_0} + e^{-t})^2 - (x - x_0)^2}\right), \end{aligned} \quad (1.2)$$

where  $(x, t) \in D_+(x_0, t_0) \cup D_-(x_0, t_0)$  and  $F(a, b; c; \zeta)$  is the hypergeometric function. (For definition of the hypergeometric function, see, e.g., [7].) When no ambiguity arises, like in (1.2), we use the notation  $x^2 := |x|^2$  for  $x \in \mathbb{R}^n$ . Thus, the function  $E$  depends on  $r^2 = (x - x_0)^2$ , that is  $E(x, t; x_0, t_0; M) = E(r, t; 0, t_0; M)$ . According to Theorem 2.12 [33], the function  $E(r, t; 0, t_0; M)$  solves the following one dimensional Klein-Gordon equation in the de Sitter space-time:

$$E_{tt}(r, t; 0, t_0; M) - e^{-2t}E_{rr}(r, t; 0, t_0; M) - M^2E(r, t; 0, t_0; M) = 0.$$

The kernels  $K_0(z, t; M)$  and  $K_1(z, t; M)$  are defined by

$$K_0(z, t; M) := -[\partial_b E(z, t; 0, b; M)]_{b=0}, \quad (1.3)$$

$$K_1(z, t; M) := E(z, t; 0, 0; M). \quad (1.4)$$

The equation (1.1) is said to be an equation with imaginary (real) mass if  $M^2 > 0$  ( $-M^2 \geq 0$ ); here  $M \in \mathbb{C}$ . Assume that  $A(x, \partial_x) = \sum_{|\alpha| \leq m} a_\alpha(x) \partial_x^\alpha$ , where  $a_\alpha(x) \in C^\infty(\Omega)$  and  $\Omega \subseteq \mathbb{R}^n$ . For the Klein-Gordon equation (1.1) we invoke the following result.

**Theorem 1.1** [33] *For  $f \in C(\Omega \times I)$ ,  $I = [0, T]$ ,  $0 < T \leq \infty$ , and  $\varphi_0, \varphi_1 \in C(\Omega)$ , let the function  $v_f(x, t; b) \in C_{x,t,b}^{m,2,0}(\Omega \times [0, 1 - e^{-T}] \times I)$  be a solution to the problem*

$$\begin{cases} v_{tt} - A(x, \partial_x)v = 0, & x \in \Omega, \quad t \in [0, 1 - e^{-T}], \\ v(x, 0; b) = f(x, b), \quad v_t(x, 0; b) = 0, & b \in I, \quad x \in \Omega, \end{cases} \quad (1.5)$$

and the function  $v_\varphi(x, t) \in C_{x,t}^{m,2}(\Omega \times [0, 1 - e^{-T}])$  be a solution of the problem

$$\begin{cases} v_{tt} - A(x, \partial_x)v = 0, & x \in \Omega, \quad t \in [0, 1 - e^{-T}], \\ v(x, 0) = \varphi(x), \quad v_t(x, 0) = 0, & x \in \Omega. \end{cases} \quad (1.6)$$

Then the function  $u = u(x, t)$  defined by

$$\begin{aligned} u(x, t) &= 2 \int_0^t db \int_0^{\phi(t)-\phi(b)} v_f(x, r; b) E(r, t; 0, b; M) dr + e^{\frac{t}{2}} v_{\varphi_0}(x, \phi(t)) \\ &\quad + 2 \int_0^{\phi(t)} v_{\varphi_0}(x, s) K_0(s, t; M) ds + 2 \int_0^{\phi(t)} v_{\varphi_1}(x, s) K_1(s, t; M) ds, \quad x \in \Omega, \quad t \in I, \end{aligned}$$

with  $\phi(t) := 1 - e^{-t}$ , solves the problem

$$\begin{cases} u_{tt} - e^{-2t} A(x, \partial_x)u - M^2 u = f, & x \in \Omega, \quad t \in I, \\ u(x, 0) = \varphi_0(x), \quad u_t(x, 0) = \varphi_1(x), & x \in \Omega. \end{cases} \quad (1.7)$$

Here the kernels  $E$ ,  $K_0$  and  $K_1$  have been defined in (1.2), (1.3) and (1.4), respectively.

We note that the operator  $A(x, \partial_x)$  is of arbitrary order, that is, the equation of (1.7) can be an evolution equation, not necessarily hyperbolic. Then, the problems (1.5) and (1.7) can be mixed initial-boundary value problems involving the boundary condition. Next, we stress that interval  $[0, 1 - e^{-T}] \subseteq [0, 1]$ , which appears in (1.5), reflects the fact that de Sitter model possesses the horizon [16]; existence of the horizon in the de Sitter model is widely used to define an asymptotically de Sitter space (see, e.g., [5, 28]) and to involve geometry into the analysis of the operators on the de Sitter space (see, e.g., [9, 23, 25, 27]).

In more explicit form the kernels can be written as follows

$$\begin{aligned} K_0(z, t; M) &:= 4^{-M} e^{tM} ((1 + e^{-t})^2 - z^2)^M \frac{1}{[(1 - e^{-t})^2 - z^2] \sqrt{(1 + e^{-t})^2 - z^2}} \\ &\quad \times \left[ (e^{-t} - 1 + M(e^{-2t} - 1 - z^2)) F\left(\frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(1 - e^{-t})^2 - z^2}{(1 + e^{-t})^2 - z^2}\right) \right. \\ &\quad \left. + (1 - e^{-2t} + z^2) \left(\frac{1}{2} + M\right) F\left(-\frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(1 - e^{-t})^2 - z^2}{(1 + e^{-t})^2 - z^2}\right) \right], \\ K_1(z, t; M) &:= 4^{-M} e^{Mt} ((1 + e^{-t})^2 - z^2)^{-\frac{1}{2}+M} F\left(\frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(1 - e^{-t})^2 - z^2}{(1 + e^{-t})^2 - z^2}\right). \end{aligned}$$

We give one example of the equation with the variable coefficients that is amenable to the integral transform method. If  $\Omega = \Pi$  is a non-Euclidean space of constant negative curvature and the equation of the problems (1.5) and (1.6) is a non-Euclidean wave equation, then the explicit representation formulas are known (see, e.g., [17, 20]) and the Huygens' principle is a consequence of those formulas. Thus, for a non-Euclidean wave equation, due to Theorem 1.1, the functions  $v_f(x, t; b)$  and  $v_\varphi(x, t)$  have explicit representations, and the arguments of [29, 31] allow us to derive for the solution  $u(x, t)$  of the problem (1.7) in the de Sitter metric with hyperbolic spatial geometry the explicit representation, the  $L^p - L^q$  estimates, and to examine the Huygens' principle.

Thus, according to Theorem 1.1 for the solution  $\psi$  of the equation

$$\psi_{tt} + n\psi_t - e^{-2t}A(x, \partial_x)\psi + m^2\psi = f, \quad (1.8)$$

due to the relation  $u = e^{\frac{n}{2}t}\psi$ , we obtain

$$\psi(x, t) = 2e^{-\frac{n}{2}t} \int_0^t db \int_0^{e^{-b}-e^{-t}} dr e^{\frac{n}{2}b} v_f(x, r; b) E(r, t; 0, b; M), \quad (1.9)$$

where the function  $v_f(x, t; b)$  is defined by (1.5), and

$$M^2 = \frac{n^2}{4} - m^2.$$

Then, for the solution  $\psi$  of the Cauchy problem

$$\psi_{tt} + n\psi_t - e^{-2t}A(x, \partial_x)\psi + m^2\psi = 0, \quad \psi(x, 0) = \psi_0(x), \quad \psi_t(x, 0) = \psi_1(x), \quad (1.10)$$

due to the relation  $u = e^{\frac{n}{2}t}\psi$ , we obtain

$$\begin{aligned} \psi(x, t) &= e^{-\frac{n-1}{2}t} v_{\psi_0}(x, \phi(t)) + e^{-\frac{n}{2}t} \int_0^1 v_{\psi_0}(x, \phi(t)s) \left( 2K_0(\phi(t)s, t; M) + nK_1(\phi(t)s, t; M) \right) \phi(t) ds \\ &\quad + 2e^{-\frac{n}{2}t} \int_0^1 v_{\psi_1}(x, \phi(t)s) K_1(\phi(t)s, t; M) \phi(t) ds, \quad t > 0. \end{aligned} \quad (1.11)$$

We stress here that the integral transform of this section allows to shrink (contract) the time interval  $[0, \infty)$  to the bounded interval  $[0, 1]$ .

Then, for the linear strictly hyperbolic equations with smooth coefficients the integral representations hold also for the functions in the Sobolev and Besov spaces.

Henceforth, we use the following classification of the physical mass  $m$ . The mass  $m$  is *large* if  $m \geq n$ , *small* if  $m < n$ , and *critical* (for the Huygens' principle) if  $m = \sqrt{n^2 - 1}/2$ .

## 2 Large Mass. $B_p^{s,q} - B_{p'}^{s',q}$ Estimates

The following estimates proved by Brenner [10] are crucial for the estimates of this and the next sections. Let  $\varphi_j = \varphi(2^{-j}\xi)$ ,  $j > 0$ , and  $\varphi_0 = 1 - \sum_{j=1}^{\infty} \varphi_j$ , where  $\varphi \in C_0^\infty(\mathbb{R}^n)$  with  $\varphi \geq 0$  and  $\text{supp } \varphi \subseteq \{\xi \in \mathbb{R}^n; 1/2 < |\xi| < 2\}$ , is that

$$\sum_{-\infty}^{\infty} \varphi(2^{-j}\xi) = 1, \quad \xi \neq 0.$$



The norm  $\|g\|_{B_p^{s,q}}$  of the Besov space  $B_p^{s,q}$  is defined as follows

$$\|v\|_{B_p^{s,q}} = \left( \sum_{j=0}^{\infty} (2^{js} \|\mathcal{F}^{-1}(\varphi_j \hat{v})\|_p)^q \right)^{1/q},$$

where  $\hat{v}$  is the Fourier transform of  $v$ .

**Theorem 2.1** (Brenner [10]) *Let  $A = A(x, D)$  be a second order negative elliptic differential operator with real  $C^\infty$ -coefficients such that  $A(x, D) = A(\infty, D)$  for  $|x|$  large enough. Let  $u(t) = G_0(t)g_0 + G_1(t)g_1$  be the solution of*

$$\partial_t^2 u - A(x, D)u = 0, \quad x \in \mathbb{R}^n, \quad t \geq 0, \quad (2.1)$$

$$u(x, 0) = g_0(x), \quad u_t(x, 0) = g_1(x), \quad x \in \mathbb{R}^n. \quad (2.2)$$

Then for each  $T < \infty$  there is a constant  $C = C(T)$  such that if  $(n+1)\delta \leq \nu + s - s'$ ,

$$\|G_\nu(t)g\|_{B_{p'}^{s',q}} \leq C(T)t^{\nu+s-s'-2n\delta} \|g\|_{B_p^{s,q}}, \quad 0 < t \leq T. \quad (2.3)$$

Here  $s, s' \geq 0$ ,  $q \geq 1$ ,  $1 \leq p \leq 2$ ,  $1/p + 1/p' = 1$ , and  $\delta = 1/p - 1/2$ .

The estimate (2.3) can be also written as

$$\|G_\nu(t)g\|_{B_{p'}^{s',q}} \leq C(T)t^{\nu+s-s'-n(1/p-1/p')} \|g\|_{B_p^{s,q}}, \quad 0 < t \leq T, \quad \nu = 0, 1.$$

We note that in this theorem  $T < \infty$ , but on the other hand, due to the integral transform of the previous section, it is possible to reduce the problem with infinite time to the problem with the finite time, and apply Theorem 2.1 with  $T = 1$  only. We note that the condition  $A(x, D) = A(\infty, D)$  for  $|x|$  large enough is not restrictive since the de Sitter space-time has permanently bounded domain of influence.

In fact, Theorem 3.2 [10] states similar result for the case of operator  $A = A(x, t, D)$  that makes possible to generalize some results of the present paper.

## 2.1 Large Mass. $B_p^{s,q} - B_{p'}^{s',q}$ Estimates for Equation without Source

The decay estimates for the energy of the solution of the Cauchy problem for the wave equation without source can be proved by the representation formula,  $L^1 - L^\infty$  and  $L^2 - L^2$  estimates, and interpolation argument. (See, e.g., [26, Theorem 2.1].) The proof of Theorem 2.1 given in [10] is based on the Fourier operators theory, microlocal consideration and dyadic decomposition of the phase space. In order to prove in the next theorem the similar result in the de Sitter space-time, we appeal to the representation formula provided by Theorem 1.1 and then apply Theorem 2.1.

**Theorem 2.2** *Let  $u(t) = G_{0,dS}(t)\varphi_0 + G_{1,dS}(t)\varphi_1$  be the solution of the Cauchy problem (1.7) with  $f = 0$ . Then the operators  $G_{0,dS}(t)$  and  $G_{1,dS}(t)$  satisfy the following estimates*

$$\|G_{0,dS}(t)\varphi_0\|_{B_{p'}^{s',q}} \leq C_M(1+t)^{1-\text{sgn}M}(1-e^{-t})^{s-s'-2n\delta} e^{\frac{t}{2}} \|\varphi_0\|_{B_p^{s,q}},$$

and

$$\|G_{1,dS}(t)\varphi_1\|_{B_{p'}^{s',q}} \leq C_M(1+t)^{1-\text{sgn}M}(1-e^{-t})^{1+s-s'-2n\delta} \|\varphi_1\|_{B_p^{s,q}},$$

for all  $t \in (0, \infty)$ , provided that  $(n+1)\delta \leq s - s'$ ,  $1 < p \leq 2$ ,  $\frac{1}{p} + \frac{1}{p'} = 1$ ,  $s - s' - 2n\delta > -1$ , and  $\delta = 1/p - 1/2$ .

They can be written as follows

$$\|G_{\nu, dS}(t)\varphi\|_{B_{p'}^{s', q}} \leq C_M e^{\frac{t}{2}(1-\nu)}(1+t)^{1-\text{sgn}M}(1-e^{-t})^{\nu+s-s'-2n\delta}\|\varphi\|_{B_p^{s, q}}, \quad \nu = 0, 1.$$

*Proof.* We start with the operator  $G_{1, dS}(t)$ . Due to Theorem 2.1 for the solution  $u = u(x, t)$  of the Cauchy problem (2.1)-(2.2) with  $\varphi_0 = 0$  and according to (2.3) we have for  $T = 1$ :

$$\|G_0(t)g\|_{B_{p'}^{s', q}} \leq C(T)t^{s-s'-2n\delta}\|g\|_{B_p^{s, q}}, \quad 0 < t \leq T, \quad (n+1)\delta \leq s - s'.$$

Then, due to Theorem 1.1

$$\begin{aligned} \|u(x, t)\|_{B_{p'}^{s', q}} &\leq 2 \int_0^{1-e^{-t}} \|v_{\varphi_1}(x, r)K_1(r, t; -iM)\|_{B_{p'}^{s', q}} dr \\ &\leq C\|\varphi_1\|_{B_p^{s, q}} \int_0^{1-e^{-t}} r^{s-s'-2n\delta} |K_1(r, t; -iM)| dr \\ &\leq C_M\|\varphi_1\|_{B_p^{s, q}} e^{-t(s-s'-2n\delta)} \int_0^{e^t-1} y^{s-s'-2n\delta} ((e^t+1)^2 - y^2)^{-\frac{1}{2}} \\ &\quad \times \left( F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(e^t-1)^2 - y^2}{(e^t+1)^2 - y^2}\right) \right)^{1-\text{sgn}M} dy. \end{aligned}$$

To continue we prove the next simple generalization of Lemma 9.2 [29].

**Lemma 2.3** Assume that  $0 \geq a > -1$ . Then

$$\int_0^{z-1} r^a \frac{1}{\sqrt{(z+1)^2 - r^2}} F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(z-1)^2 - r^2}{(z+1)^2 - r^2}\right) dr \leq Cz^{-1}(z-1)^{1+a}(1 + \ln z),$$

for all  $z > 1$ .

*Proof.* If  $1 < z \leq M$  with some constant  $M$ , then the argument of the hypergeometric functions is bounded,

$$\frac{(z-1)^2 - r^2}{(z+1)^2 - r^2} \leq \frac{(z-1)^2}{(z+1)^2} \leq \frac{(M-1)^2}{(M+1)^2} < 1 \quad \text{for all } r \in (0, z-1), \quad (2.4)$$

and

$$\begin{aligned} &\int_0^{z-1} r^a \frac{1}{\sqrt{(z+1)^2 - r^2}} F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(z-1)^2 - r^2}{(z+1)^2 - r^2}\right) dr \\ &\leq C_M(z-1)^{1+a}, \quad \text{for all } 1 < z \leq M. \end{aligned}$$

Hence, we can restrict ourselves to the case of large  $z$ , that is  $z \geq M$ . In particular, we choose  $M > 6$  and split integral into two parts:

$$\begin{aligned} &\int_0^{z-1} r^a \frac{1}{\sqrt{(z+1)^2 - r^2}} F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(z-1)^2 - r^2}{(z+1)^2 - r^2}\right) dr \\ &= \int_0^{\sqrt{(z+1)^2 - 8z}} r^a \frac{1}{\sqrt{(z+1)^2 - r^2}} F\left(\frac{1}{2}, \frac{1}{2}; 1; 1 - \frac{4z}{(z+1)^2 - r^2}\right) dr \\ &\quad + \int_{\sqrt{(z+1)^2 - 8z}}^{z-1} r^a \frac{1}{\sqrt{(z+1)^2 - r^2}} F\left(\frac{1}{2}, \frac{1}{2}; 1; 1 - \frac{4z}{(z+1)^2 - r^2}\right) dr. \end{aligned}$$

For the second part we have  $z \geq M > 6$  and  $r \geq \sqrt{(z+1)^2 - 8z}$ , then

$$\frac{4z}{(z+1)^2 - r^2} \geq \frac{1}{2} \implies 0 < 1 - \frac{4z}{(z+1)^2 - r^2} \leq \frac{1}{2} \quad (2.5)$$

for such  $r$  and  $z$  implies

$$\left| F\left(\frac{1}{2}, \frac{1}{2}; 1; 1 - \frac{4z}{(z+1)^2 - r^2}\right) \right| \leq C. \quad (2.6)$$

Hence, (2.5) and (2.6) imply

$$\begin{aligned} & \int_{\sqrt{(z+1)^2 - 8z}}^{z-1} r^a \frac{1}{\sqrt{(z+1)^2 - r^2}} F\left(\frac{1}{2}, \frac{1}{2}; 1; 1 - \frac{4z}{(z+1)^2 - r^2}\right) dr \\ & \leq C \int_0^{z-1} r^a \frac{1}{\sqrt{(z+1)^2 - r^2}} dr \\ & \leq C(1+z)^a \quad \text{for all } z \geq M > 6. \end{aligned}$$

For the first integral  $r \leq \sqrt{(z+1)^2 - 8z}$  and  $z \geq M > 6$  imply  $8z \leq (z+1)^2 - r^2$ . It follows

$$\left| F\left(\frac{1}{2}, \frac{1}{2}; 1; 1 - \frac{4z}{(z+1)^2 - r^2}\right) \right| \leq C \left| \ln\left(\frac{4z}{(z+1)^2 - r^2}\right) \right| \leq C(1 + \ln z).$$

Then we obtain

$$\begin{aligned} & \int_0^{\sqrt{(z+1)^2 - 8z}} r^a \frac{1}{\sqrt{(z+1)^2 - r^2}} F\left(\frac{1}{2}, \frac{1}{2}; 1; 1 - \frac{4z}{(z+1)^2 - r^2}\right) dr \\ & \leq C(1 + \ln z) \int_0^{z-1} r^a \frac{1}{\sqrt{(z+1)^2 - r^2}} dr \\ & \leq C(1 + \ln z)(1+z)^a. \end{aligned}$$

Lemma is proved.  $\square$

In order to complete the proof of the theorem for this case, with  $z = e^t$  and  $s - s' - 2n\delta > -1$  we conclude

$$\begin{aligned} & \|u(x, t)\|_{B_{p'}^{s', q}} \\ & \leq C_M \|\varphi_1\|_{B_p^{s, q}} e^{-t(s-s'-2n\delta)} \\ & \quad \times \int_0^{e^t-1} y^{s-s'-2n\delta} ((e^t+1)^2 - y^2)^{-\frac{1}{2}} \left( F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(e^t-1)^2 - y^2}{(e^t+1)^2 - y^2}\right) \right)^{1-\text{sgn}M} dy \\ & \leq C_M \|\varphi_1\|_{B_p^{s, q}} e^{-t(s-s'-2n\delta)} e^{-t} (e^t-1)^{1+s-s'-2n\delta} (1+t)^{1-\text{sgn}M}. \end{aligned}$$

Thus, in the case of  $\varphi_0 = 0$  the theorem is proved.

Next we turn to the operator  $G_{0,dS}(t)$ . Due to Theorem 2.1 for the solution  $u = u(x, t)$  of the Cauchy problem (2.1)-(2.2) with  $\varphi_1 = 0$  and to Theorem 1.1 we have:

$$\begin{aligned} \|u(x, t)\|_{B_{p'}^{s', q}} & \leq e^{t/2} \|v_{\varphi_0}(x, 1 - e^{-t})\|_{B_{p'}^{s', q}} + 2 \int_0^{1-e^{-t}} \|v_{\varphi_0}(x, r) K_0(r, t; -iM)\|_{B_{p'}^{s', q}} dr \\ & \leq e^{t/2} \|v_{\varphi_0}(x, 1 - e^{-t})\|_{B_{p'}^{s', q}} + C \|\varphi_0\|_{B_p^{s, q}} \int_0^{1-e^{-t}} r^{s-s'-2n\delta} |K_0(r, t; -iM)| dr \\ & \leq C \left( e^{\frac{t}{2}} (1 - e^{-t})^{s-s'-2n\delta} + \int_0^{1-e^{-t}} r^{s-s'-2n\delta} |K_0(r, t; -iM)| dr \right) \|\varphi_0(x)\|_{B_p^{s, q}}. \end{aligned}$$

One can estimate the last integral

$$\begin{aligned}
& \int_0^{1-e^{-t}} r^{s-s'-2n\delta} |K_0(r, t; -iM)| dr \\
& \leq e^{-t(s-s'-2n\delta)} \int_0^{e^t-1} y^{s-s'-2n\delta} \frac{1}{[(e^t-1)^2 - y^2] \sqrt{(e^t+1)^2 - y^2}} \\
& \quad \times \left| (e^t - e^{2t} - iM(1 - e^{2t} - y^2)) F\left(\frac{1}{2} + iM, \frac{1}{2} + iM; 1; \frac{(e^t-1)^2 - y^2}{(e^t+1)^2 - y^2}\right) \right. \\
& \quad \left. + (e^{2t} - 1 + y^2) \left(\frac{1}{2} - iM\right) F\left(-\frac{1}{2} + iM, \frac{1}{2} + iM; 1; \frac{(e^t-1)^2 - y^2}{(e^t+1)^2 - y^2}\right) \right| dy.
\end{aligned}$$

The following proposition with  $a = s - s' - 2n\delta$  gives the remaining estimate for that integral and allow us to complete the proof of the theorem.

**Proposition 2.4** *If  $a > -1$ , then*

$$\begin{aligned}
& \int_0^{z-1} y^a \frac{1}{[(z-1)^2 - y^2] \sqrt{(z+1)^2 - y^2}} \\
& \times \left| (z - z^2 - iM(1 - z^2 - y^2)) F\left(\frac{1}{2} + iM, \frac{1}{2} + iM; 1; \frac{(z-1)^2 - y^2}{(z+1)^2 - y^2}\right) \right. \\
& \quad \left. + (z^2 - 1 + y^2) \left(\frac{1}{2} - iM\right) F\left(-\frac{1}{2} + iM, \frac{1}{2} + iM; 1; \frac{(z-1)^2 - y^2}{(z+1)^2 - y^2}\right) \right| dy \\
& \leq C_M z^{-\frac{1}{2}} (z-1)^{1+a} (1 + \ln z)^{1-\text{sgn}M} \quad \text{for all } z > 1.
\end{aligned}$$

*Proof.* (Comp. with Prop.10.2 [29].) We follow the arguments have been used in the proof of Proposition 8.3 [29]. If  $1 \leq z \leq N$  with some constant  $N$ , then the argument of the hypergeometric functions is bounded (2.4), and the integral can be estimated by:

$$\begin{aligned}
& \int_0^{z-1} y^a \frac{1}{[(z-1)^2 - y^2] \sqrt{(z+1)^2 - y^2}} \\
& \times \left| (z - z^2 - iM(1 - z^2 - y^2)) F\left(\frac{1}{2} + iM, \frac{1}{2} + iM; 1; \frac{(z-1)^2 - y^2}{(z+1)^2 - y^2}\right) \right. \\
& \quad \left. + (z^2 - 1 + y^2) \left(\frac{1}{2} - iM\right) F\left(-\frac{1}{2} + iM, \frac{1}{2} + iM; 1; \frac{(z-1)^2 - y^2}{(z+1)^2 - y^2}\right) \right| dy \\
& \leq C_M \int_0^{z-1} y^a \left[ \frac{1}{\sqrt{(z+1)^2 - y^2}} \left\{ 1 + z^2 \frac{1}{(z+1)^2 - y^2} \right\} \right] dy \\
& \leq C_M z^{-1} (z-1)^{1+a} \quad \text{for all } z \in [1, N].
\end{aligned}$$

Thus, we can restrict ourselves to the case of large  $z \geq N$  in both zones  $Z_1(\varepsilon, z)$  and  $Z_2(\varepsilon, z)$ , which are defined by

$$\begin{aligned}
Z_1(\varepsilon, z) &:= \left\{ (z, r) \mid \frac{(z-1)^2 - r^2}{(z+1)^2 - r^2} \leq \varepsilon, \ 0 \leq r \leq z-1 \right\}, \\
Z_2(\varepsilon, z) &:= \left\{ (z, r) \mid \varepsilon \leq \frac{(z-1)^2 - r^2}{(z+1)^2 - r^2}, \ 0 \leq r \leq z-1 \right\},
\end{aligned}$$

respectively. In the first zone we have (7.11) [29]:

$$\begin{aligned}
& \left| (z - z^2 - iM(1 - z^2 - r^2))F\left(\frac{1}{2} + iM, \frac{1}{2} + iM; 1; \frac{(z-1)^2 - r^2}{(z+1)^2 - r^2}\right) \right. \\
& \quad \left. + (z^2 - 1 + r^2)\left(\frac{1}{2} - iM\right)F\left(-\frac{1}{2} + iM, \frac{1}{2} + iM; 1; \frac{(z-1)^2 - r^2}{(z+1)^2 - r^2}\right) \right| \\
& \leq \frac{1}{2}[(z-1)^2 - r^2] + \frac{1}{8} \frac{(z-1)^2 - r^2}{(z+1)^2 - r^2} \\
& \quad \times |(1 - 2iM)(-1 + 4M^2)(r^2 + z^2 - 1) + 2(1 + 2iM)^2(-z^2 + z + iM(r^2 + z^2 - 1))| \\
& \quad + \left( |z - z^2 - iM(1 - z^2 - r^2)| + |z^2 - 1 + r^2| \right) O\left(\left(\frac{(z-1)^2 - r^2}{(z+1)^2 - r^2}\right)^2\right).
\end{aligned}$$

Consider therefore the following three estimates. For the first one we have (comp. with Prop.10.2[29])

$$A_8 := \int_{(z,r) \in Z_1(\varepsilon, z)} r^a \frac{1}{\sqrt{(z+1)^2 - r^2}} dr \leq C z^{-1} (z-1)^{1+a} \quad \text{for all } z \in [N, \infty).$$

For  $0 \geq a > -1$  and  $z \geq N$  the following integral can be easily estimated:

$$\begin{aligned}
\int_0^{z-1} r^a \frac{1}{((z+1)^2 - r^2)^{3/2}} dr &= \int_0^{z/2} r^a \frac{1}{((z+1)^2 - r^2)^{3/2}} dr + \int_{z/2}^{z-1} r^a \frac{1}{((z+1)^2 - r^2)^{3/2}} dr \\
&\leq \frac{16}{9} z^{-3} \int_0^{z/2} r^a dr + \frac{z^a}{4^a} \int_{z/2}^{z-1} \frac{1}{((z+1)^2 - r^2)^{3/2}} dr \\
&\leq C z^{a-3/2} \quad \text{for all } z \in [N, \infty).
\end{aligned}$$

Hence,

$$\begin{aligned}
A_9 &:= z^2 \int_{(z,r) \in Z_1(\varepsilon, z)} r^a \frac{1}{\sqrt{(z+1)^2 - r^2}} \frac{1}{(z+1)^2 - r^2} dr \\
&\leq z^2 \int_0^{z-1} r^a \frac{1}{\sqrt{(z+1)^2 - r^2}} \frac{1}{(z+1)^2 - r^2} dr \\
&\leq C z^{-\frac{1}{2}} (z-1)^{1+a} \quad \text{for all } z \in [N, \infty),
\end{aligned}$$

and

$$\begin{aligned}
A_{10} &:= z^2 \int_{(z,r) \in Z_1(\varepsilon, z)} r^a \frac{1}{((z-1)^2 - r^2) \sqrt{(z+1)^2 - r^2}} \left( \frac{(z-1)^2 - r^2}{(z+1)^2 - r^2} \right)^2 dr \\
&\leq z^2 \int_{(z,r) \in Z_1(\varepsilon, z)} r^a \frac{1}{\sqrt{(z+1)^2 - r^2}} \frac{1}{(z+1)^2 - r^2} dr \\
&\leq C z^{-\frac{1}{2}} (z-1)^{1+a} \quad \text{for all } z \in [N, \infty).
\end{aligned}$$

Finally,

$$\int_{(z,y) \in Z_1(\varepsilon, z)} y^a \frac{1}{[(z-1)^2 - y^2] \sqrt{(z+1)^2 - y^2}}$$

$$\begin{aligned}
& \times \left| (z - z^2 - iM(1 - z^2 - y^2)) F\left(\frac{1}{2} + iM, \frac{1}{2} + iM; 1; \frac{(z-1)^2 - y^2}{(z+1)^2 - y^2}\right) \right. \\
& \quad \left. + (z^2 - 1 + y^2) \left(\frac{1}{2} - iM\right) F\left(-\frac{1}{2} + iM, \frac{1}{2} + iM; 1; \frac{(z-1)^2 - y^2}{(z+1)^2 - y^2}\right) \right| dy \\
& \leq C z^{-\frac{1}{2}} (z-1)^{1+a} \quad \text{for all } z \in [1, \infty).
\end{aligned}$$

In the second zone we use (7.12)-(7.14)[29] :

$$\varepsilon \leq \frac{(z-1)^2 - r^2}{(z+1)^2 - r^2} \leq 1 \quad \text{and} \quad \frac{1}{(z-1)^2 - r^2} \leq \frac{1}{\varepsilon[(z+1)^2 - r^2]}. \quad (2.7)$$

According to (7.3) [29]

$$\left| F\left(\frac{1}{2} + iM, \frac{1}{2} + iM; 1; z\right) \right| \leq C_M (1 - \ln(1 - z))^{1 - \text{sgn} M} \quad \text{for all } z \in [0, 1).$$

Thus, the hypergeometric functions obey the estimates

$$\left| F\left(-\frac{1}{2} + iM, \frac{1}{2} + iM; 1; z\right) \right|, \left| F\left(\frac{1}{2} + iM, \frac{1}{2} + iM; 1; z\right) \right| \leq C_M \quad \text{for all } z \in [\varepsilon, 1), \quad (2.8)$$

and

$$\left| F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(z-1)^2 - r^2}{(z+1)^2 - r^2}\right) \right| \leq C (1 + \ln z), \quad \text{for all } (z, r) \in Z_2(\varepsilon, z). \quad (2.9)$$

In the second zone we use (2.7), (2.8), and (2.9). Thus, we have to estimate the next two integrals:

$$\begin{aligned}
A_{11} &:= z^2 \int_{(z,r) \in Z_2(\varepsilon, z)} r^a \frac{1}{((z-1)^2 - r^2) \sqrt{(z+1)^2 - r^2}} dr, \\
A_{12} &:= z^2 (1 + \ln z)^{1 - \text{sgn} M} \int_{(z,r) \in Z_2(\varepsilon, z)} r^a \frac{1}{((z-1)^2 - r^2) \sqrt{(z+1)^2 - r^2}} dr.
\end{aligned}$$

We apply (2.7) to  $A_{11}$  and obtain

$$A_{11} \leq C_\varepsilon z^2 \int_{(z,r) \in Z_2(\varepsilon, z)} r^a \frac{1}{[(z+1)^2 - r^2]} \frac{1}{\sqrt{(z+1)^2 - r^2}} dr \leq C_\varepsilon z^{-\frac{1}{2}} (z-1)^{1+a}$$

for all  $z \in [1, \infty)$ , while

$$A_{12} \leq C_\varepsilon z^{-\frac{1}{2}} (z-1)^{1+a} (1 + \ln z)^{1 - \text{sgn} M} \quad \text{for all } z \in [1, \infty).$$

The proposition is proved.  $\square$

To complete the proof of the theorem we write

$$\begin{aligned}
& \int_0^{1-e^{-t}} r^{s-s'-2n\delta} |K_0(r, t; -iM)| dr \\
& \leq e^{-t[s-s'-2n\delta]} \int_0^{e^t-1} y^{s-s'-2n\delta} \frac{1}{[(e^t-1)^2 - y^2] \sqrt{(e^t+1)^2 - y^2}} \\
& \quad \times \left| (e^t - e^{2t} - iM(1 - e^{2t} - y^2)) F\left(\frac{1}{2} + iM, \frac{1}{2} + iM; 1; \frac{(e^t-1)^2 - y^2}{(e^t+1)^2 - y^2}\right) \right. \\
& \quad \left. + (e^{2t} - 1 + y^2) \left(\frac{1}{2} - iM\right) F\left(-\frac{1}{2} + iM, \frac{1}{2} + iM; 1; \frac{(e^t-1)^2 - y^2}{(e^t+1)^2 - y^2}\right) \right| dy \\
& \leq C e^{-t[\frac{1}{2} + s-s'-2n\delta]} (e^t - 1)^{1+s-s'-2n\delta} (1+t)^{1 - \text{sgn} M}.
\end{aligned}$$

Thus,

$$\begin{aligned}
& \|u(x, t)\|_{B_{p'}^{s', q}} \\
& \leq C \left( e^{\frac{t}{2}} (1 - e^{-t})^{s-s'-2n\delta} + \int_0^{1-e^{-t}} r^{s-s'-2n\delta} |K_0(r, t; -iM)| dr \right) \|\varphi_0(x)\|_{B_p^{s, q}} \\
& \leq C \left( e^{\frac{t}{2}} (1 - e^{-t})^{s-s'-2n\delta} + e^{-t[\frac{1}{2}+s-s'-2n\delta]} (e^t - 1)^{1+s-s'-2n\delta} (1+t)^{1-\text{sgn}M} \right) \|\varphi_0(x)\|_{B_p^{s, q}} \\
& \leq C(1+t)^{1-\text{sgn}M} e^{\frac{t}{2}} (1 - e^{-t})^{s-s'-2n\delta} \|\varphi_0(x)\|_{B_p^{s, q}}.
\end{aligned}$$

Theorem is proved.  $\square$

## 2.2 Large Mass. $B_p^{s, q} - B_{p'}^{s', q}$ Estimates for the Equation with Source

In general, the Duhamel's principle allows us to reduce the case with a source term to the case of the Cauchy problem without source term and consequently to derive the decay estimates for the equation. For the equation under consideration we appeal to the representation formula of Theorem 1.1. In this subsection we consider the Cauchy problem for the equation with the source term and with zero initial data.

**Theorem 2.5** *Let  $u = u(x, t)$  be solution of the Cauchy problem*

$$\begin{cases} u_{tt} - e^{-2t} A(x, \partial_x)u - M^2 u = f, & x \in \mathbb{R}^n, \ t > 0, \\ u(x, 0) = 0, \quad u_t(x, 0) = 0, & x \in \mathbb{R}^n. \end{cases} \quad (2.10)$$

*Then for  $n \geq 2$  one has the following estimate*

$$\begin{aligned}
& \|u(x, t)\|_{B_{p'}^{s', q}} \\
& \leq C_M \int_0^t db \|f(x, b)\|_{B_p^{s, q}} \int_0^{e^{-b}-e^{-t}} r^{s-s'-2n\delta} \frac{\left( F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(e^{-b}-e^{-t})^2-r^2}{(e^{-b}+e^{-t})^2-r^2}\right) \right)^{1-\text{sgn}M}}{\sqrt{(e^{-t}+e^{-b})^2-r^2}} dr
\end{aligned}$$

for all  $t > 0$ , provided that  $1 < p \leq 2$ ,  $\frac{1}{p} + \frac{1}{p'} = 1$ ,  $s - s' - 2n\delta > -1$ ,  $s, s' \geq 0$ ,  $(n+1)\delta \leq s - s'$ , and  $\delta = 1/p - 1/2$ .

*Proof.* We use the representation given by Theorem 1.1. Due to Theorem 2.1 for the equation of (2.10), we have

$$\|u(x, t)\|_{B_{p'}^{s', q}} \leq 2 \int_0^t db \int_0^{\phi(t)-\phi(b)} \|v_f(x, r; b)\|_{B_{p'}^{s', q}} |E(r, t; 0, b; -iM)| dr,$$

where  $\phi(t) := 1 - e^{-t}$ , and, consequently,

$$\begin{aligned}
& \|u(x, t)\|_{B_{p'}^{s', q}} \\
& \leq C_M \int_0^t db \int_0^{e^{-b}-e^{-t}} \|v_f(x, r; b)\|_{B_{p'}^{s', q}} \frac{\left( F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(e^{-b}-e^{-t})^2-r^2}{(e^{-b}+e^{-t})^2-r^2}\right) \right)^{1-\text{sgn}M}}{\sqrt{(e^{-t}+e^{-b})^2-r^2}} dr \\
& \leq C_M \int_0^t db \|f(x, b)\|_{B_p^{s, q}} \int_0^{e^{-b}-e^{-t}} r^{s-s'-2n\delta} \frac{\left( F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(e^{-b}-e^{-t})^2-r^2}{(e^{-b}+e^{-t})^2-r^2}\right) \right)^{1-\text{sgn}M}}{\sqrt{(e^{-t}+e^{-b})^2-r^2}} dr.
\end{aligned}$$

Theorem is proved.  $\square$

We are going to transform the estimate of the last theorem to a more compact form. To this aim we estimate for  $s - s' - 2n\delta > -1$  the last integral of the right hand side. If we replace  $e^{-(b-t)}$  with  $z := e^{-(b-t)} > 1$ , then the integral will be simplified:

$$\begin{aligned} & \int_0^{e^{-b}-e^{-t}} r^{s-s'-2n\delta} \frac{1}{\sqrt{(e^{-t}+e^{-b})^2-r^2}} \left( F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(e^{-b}-e^{-t})^2-r^2}{(e^{-b}+e^{-t})^2-r^2}\right) \right)^{1-\text{sgn}M} dr \\ &= e^{-t(s-s'-2n\delta)} \int_0^{z-1} y^{s-s'-2n\delta} \frac{1}{\sqrt{(z+1)^2-y^2}} \left( F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(z-1)^2-y^2}{(z+1)^2-y^2}\right) \right)^{1-\text{sgn}M} dy \end{aligned}$$

Then we apply Lemma 2.3 and obtain the following corollary.

**Corollary 2.6** *Let  $u = u(x, t)$  be a solution of the Cauchy problem (2.10). Then for  $n \geq 2$  one has the following estimate*

$$\|u(x, t)\|_{B_{p'}^{s', q}} \leq C_M \int_0^t db \|f(x, b)\|_{B_p^{s, q}} e^b \left( e^{-b} - e^{-t} \right)^{1+s-s'-2n\delta} (1+t-b)^{1-\text{sgn}M} db,$$

provided that  $1 < p \leq 2$ ,  $\frac{1}{p} + \frac{1}{p'} = 1$ ,  $s, s' \geq 0$ ,  $s - s' - 2n\delta > -1$ ,  $(n+1)\delta \leq s - s'$ .

*Proof.* Indeed, we apply Lemma 2.3 with  $z = e^{t-b}$  to the right-hand side of the estimate given by Theorem 2.5 :

$$\begin{aligned} & \|u(x, t)\|_{B_{p'}^{s', q}} \\ & \leq C_M \int_0^t db \|f(x, b)\|_{B_p^{s, q}} \int_0^{e^{-b}-e^{-t}} r^{s-s'-2n\delta} \frac{\left( F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(e^{-b}-e^{-t})^2-r^2}{(e^{-b}+e^{-t})^2-r^2}\right) \right)^{1-\text{sgn}M}}{\sqrt{(e^{-t}+e^{-b})^2-r^2}} dr \\ & \leq C_M \int_0^t db \|f(x, b)\|_{B_p^{s, q}} e^{-t(s-s'-2n\delta)} z^{-1} (z-1)^{1+s-s'-2n\delta} (1+\ln z)^{1-\text{sgn}M} \\ & \leq C_M \int_0^t \|f(x, b)\|_{B_p^{s, q}} e^b \left( e^{-b} - e^{-t} \right)^{1+s-s'-2n\delta} (1+t-b)^{1-\text{sgn}M} db. \end{aligned}$$

The corollary is proved.  $\square$

### 2.3 Large mass. $B_p^{s, q} - B_{p'}^{s', q}$ estimates for the covariant equation

Then, for the solution  $\psi$  of the Cauchy problem

$$\psi_{tt} + n\psi_t - e^{-2t}A(x, \partial_x)\psi + m^2\psi = 0, \quad \psi(x, 0) = \psi_0(x), \quad \psi_t(x, 0) = \psi_1(x), \quad (2.11)$$

due to the relation  $u = e^{\frac{n}{2}t}\psi$ , we obtain (1.11).

**Estimates for equation with source.** Let  $u = u(x, t)$  be a solution of the Cauchy problem (2.10). Then according to Corollary 2.6, for the solution  $\psi$  of the equation (1.8), due to (1.9), we obtain

$$\|\psi(x, t)\|_{B_{p'}^{s', q}} \leq C_M e^{-\frac{n}{2}t} \int_0^t e^{\frac{n}{2}b} \|f(x, b)\|_{B_p^{s, q}} e^b \left( e^{-b} - e^{-t} \right)^{1+s-s'-2n\delta} (1+t-b)^{1-\text{sgn}M} db. \quad (2.12)$$



For  $M > 0$  we obtain

$$\|\psi(x, t)\|_{B_{p'}^{s', q}} \leq C_M e^{-\frac{n}{2}t} \int_0^t e^{\frac{n}{2}b} \|f(x, b)\|_{B_p^{s, q}} e^b (e^{-b} - e^{-t})^{1+s-s'-2n\delta} db,$$

while for  $M = 0$  we obtain

$$\|\psi(x, t)\|_{B_{p'}^{s', q}} \leq C e^{-\frac{n}{2}t} \int_0^t e^{\frac{n}{2}b} \|f(x, b)\|_{B_p^{s, q}} e^b (e^{-b} - e^{-t})^{1+s-s'-2n\delta} (1+t-b) db.$$

In particular, for  $s = s'$ ,  $\delta = 0$  and  $p = p' = 2$ , that is for the Sobolev space  $H_{(s)}(\mathbb{R}^n)$  we have

$$\|\psi(x, t)\|_{H_{(s)}(\mathbb{R}^n)} \leq C_M e^{-\frac{n}{2}t} \int_0^t e^{\frac{n}{2}b} \|f(x, b)\|_{H_{(s)}(\mathbb{R}^n)} (1+t-b)^{1-\text{sgn}M} db. \quad (2.13)$$

Here the rates of exponential factors are independent of the curved mass  $\mathcal{M}$  if  $\mathcal{M} \neq 0$  and, consequently, of the mass  $m$ .

**Equation without source.** According to Theorem 2.2 the solution  $u = u(x, t)$  of the Cauchy problem (1.7) with  $f = 0$  satisfies the following estimate

$$\|u(x, t)\|_{B_{p'}^{s', q}} \leq C_M (1+t)^{1-\text{sgn}M} (1-e^{-t})^{s-s'-2n\delta} e^{\frac{t}{2}} \|\varphi_0(x)\|_{B_p^{s, q}},$$

if  $(n+1)\delta \leq s - s'$ , and

$$\|u(x, t)\|_{B_{p'}^{s', q}} \leq C_M (1+t)^{1-\text{sgn}M} (1-e^{-t})^{1+s-s'-2n\delta} \|\varphi_1\|_{B_p^{s, q}},$$

if  $(n+1)\delta \leq s - s'$  and  $s - s' - 2n\delta > -1$

Thus, for the solution  $\psi$  of the Cauchy problem (2.11), due to the relation  $u = e^{\frac{n}{2}t}\psi$ , we obtain the decay estimate

$$\|\psi(x, t)\|_{B_{p'}^{s', q}} \leq C_M e^{-\frac{n}{2}t} (1+t)^{1-\text{sgn}M} (1-e^{-t})^{s-s'-2n\delta} \left\{ e^{\frac{t}{2}} \|\psi_0(x)\|_{B_p^{s, q}} + (1-e^{-t}) \|\psi_1\|_{B_p^{s, q}} \right\} \quad (2.14)$$

for all  $t > 0$ , while

$$\|\psi(x, t)\|_{B_{p'}^{s', q}} \leq C_M e^{-\frac{n}{2}t} (1+t)^{1-\text{sgn}M} \left\{ e^{\frac{t}{2}} \|\psi_0(x)\|_{B_p^{s, q}} + \|\psi_1\|_{B_p^{s, q}} \right\}$$

for large  $t$ . Here the rate of decay is independent of the curved mass  $\mathcal{M}$  if  $\mathcal{M} \neq 0$  and, consequently, of the mass  $m$ .

### 3 Small Mass. $B_p^{s, q} - B_{p'}^{s', q}$ Estimates

**Equation without source.** We derive the estimates of solutions of the covariant Klein-Gordon equation from the estimates of solutions of the non-covariant equation with the imaginary mass.

**Theorem 3.1** *Let  $\psi = \psi(x, t)$  be a solution of the Cauchy problem*

$$\psi_{tt} + n\psi_t - e^{-2t}A(x, \partial_x)\psi \pm m^2\psi = 0, \quad \psi(x, 0) = \psi_0(x), \quad \psi_t(x, 0) = \psi_1(x).$$

Define  $M = \sqrt{\frac{n^2}{4} - m^2} > 0$  for the case of “plus”, and  $M = \sqrt{\frac{n^2}{4} + m^2}$  for the case of “minus”. Then, in the case of “plus” with  $m \in (0, \sqrt{n^2 - 1}/2)$  and in the case of “minus” for all  $m > 0$ , it satisfies the following  $B_p^{s,q} - B_{p'}^{s',q}$  estimate

$$\|\psi(x, t)\|_{B_{p'}^{s',q}} \leq C_{m,n,p,s}(1 - e^{-t})^{s-s'-2n\delta} e^{(M-\frac{n}{2})t} \left\{ \|\psi_0\|_{B_p^{s,q}} + (1 - e^{-t}) \|\psi_1\|_{B_p^{s,q}} \right\}$$

for all  $t \in (0, \infty)$ , provided that  $s, s' \geq 0$ ,  $(n+1)\delta \leq s - s'$ ,  $\delta = 1/p - 1/2$ , and  $1 < p \leq 2$ ,  $\frac{1}{p} + \frac{1}{p'} = 1$ . The above estimate holds also if  $m \in (\sqrt{n^2 - 1}/2, n/2)$  and  $\psi_0 = 0$  that is,

$$\|\psi(x, t)\|_{B_{p'}^{s',q}} \leq C_{m,n,p,s}(1 - e^{-t})^{1+s-s'-2n\delta} e^{(M-\frac{n}{2})t} (1 - e^{-t}) \|\psi_1\|_{B_p^{s,q}}.$$

*Proof.* First we consider the case of  $\psi_1 = 0$ . Then

$$\begin{aligned} \psi(x, t) &= e^{-\frac{n-1}{2}t} v_{\psi_0}(x, \phi(t)) \\ &\quad + e^{-\frac{n}{2}t} \int_0^1 v_{\psi_0}(x, \phi(t)\tau) (2K_0(\phi(t)\tau, t; M) + nK_1(\phi(t)\tau, t; M)) \phi(t) d\tau \end{aligned}$$

and, consequently,

$$\begin{aligned} \|\psi(x, t)\|_{B_{p'}^{s',q}} &\leq e^{-\frac{n-1}{2}t} \|v_{\psi_0}(x, \phi(t))\|_{B_{p'}^{s',q}} \\ &\quad + e^{-\frac{n}{2}t} \int_0^1 \|v_{\psi_0}(x, \phi(t)\tau)\|_{B_{p'}^{s',q}} |2K_0(\phi(t)\tau, t; M) + nK_1(\phi(t)\tau, t; M)| \phi(t) d\tau. \end{aligned} \quad (3.1)$$

If  $n \geq 2$ , then for the solution  $v = v(x, t)$  of the Cauchy problem (2.1)-(2.2) with  $\psi(x) \in C_0^\infty(\mathbb{R}^n)$  we apply Theorem 2.1

$$\|v_\psi(x, t)\|_{B_{p'}^{s',q}} \leq C t^{s-s'-2n\delta} \|\psi\|_{B_p^{s,q}} \quad \text{for all } t > 0,$$

provided that  $s, s' \geq 0$ ,  $(n+1)\delta \leq s - s'$ ,  $1 < p \leq 2$ ,  $\frac{1}{p} + \frac{1}{p'} = 1$ . Hence,

$$\|v_{\psi_0}(x, \phi(t))\|_{B_{p'}^{s',q}} \leq C \phi(t)^{s-s'-2n\delta} \|\psi_0\|_{B_p^{s,q}} \quad \text{for all } t > 0,$$

where  $\phi(t) = 1 - e^{-t}$ . Consequently, for the first term of the right-hand side of (3.1) we have

$$e^{-\frac{n-1}{2}t} \|v_{\psi_0}(x, \phi(t))\|_{B_{p'}^{s',q}} \leq C e^{-\frac{n-1}{2}t} (1 - e^{-t})^{s-s'-2n\delta} \|\psi_0\|_{B_p^{s,q}} \quad \text{for all } t > 0,$$

while for the second term we obtain

$$\begin{aligned} &e^{-\frac{n}{2}t} \int_0^1 \|v_{\psi_0}(x, \phi(t)\tau)\|_{B_{p'}^{s',q}} |2K_0(\phi(t)\tau, t; M) + nK_1(\phi(t)\tau, t; M)| \phi(t) d\tau \\ &\leq \|\psi_0\|_{B_p^{s,q}} e^{-\frac{n}{2}t} \int_0^1 \phi(t)^{s-s'-2n\delta} \tau^{s-s'-2n\delta} (|2K_0(\phi(t)\tau, t; M)| + n|K_1(\phi(t)\tau, t; M)|) \phi(t) d\tau. \end{aligned}$$

We have to estimate the following two integrals of the last inequality:

$$\int_0^1 \phi(t)^{s-s'-2n\delta} \tau^{s-s'-2n\delta} |K_0(\phi(t)\tau, t; M)| \phi(t) d\tau$$

and

$$\int_0^1 \phi(t)^{s-s'-2n\delta} \tau^{s-s'-2n\delta} |K_1(\phi(t)\tau, t; M)| \phi(t) d\tau,$$

where  $\phi(t) = 1 - e^{-t}$  and  $t > 0$ . We apply Lemma 2.3 [34] and Lemma 2.4 [34] (it is important here that  $M > 1/2$ ) in the case of  $a = s - s' - 2n\delta$  and arrive at the following estimates

$$\begin{aligned} & \int_0^1 \phi(t)^{s-s'-2n\delta} \tau^{s-s'-2n\delta} |K_0(\phi(t)\tau, t; M)| \phi(t) d\tau \\ & \leq C_{M,n,p,q,s} e^{-Mt-(s-s'-2n\delta)t} (e^t - 1)^{1+s-s'-2n\delta} (e^t + 1)^{2M-1} \quad \text{for all } t > 0, \end{aligned}$$

and for  $M > 1/2$ , while

$$\begin{aligned} & \int_0^1 \phi(t)^{s-s'-2n\delta} \tau^{s-s'-2n\delta} |K_1(\phi(t)\tau, t; M)| \phi(t) d\tau \\ & \leq C_{M,n,p,q,s} e^{-Mt-t(s-s'-2n\delta)} (e^t - 1)^{1+s-s'-2n\delta} (e^t + 1)^{2M-1} \quad \text{for all } t > 0, \end{aligned}$$

and for  $M > 0$ .

Now consider the case of  $\psi_0 = 0$ . We have

$$\psi(x, t) = 2e^{-\frac{n}{2}t} \int_0^1 v_{\psi_1}(x, \phi(t)\tau) K_1(\phi(t)\tau, t; M) \phi(t) d\tau, \quad x \in \mathbb{R}^n, \quad t > 0.$$

Then, according to Lemma 2.3 [34] we have

$$\begin{aligned} \|\psi(x, t)\|_{B_p^{s',q}} & \leq 2e^{-\frac{n}{2}t} \int_0^1 \|v_{\psi_1}(x, \phi(t)\tau)\|_{B_p^{s',q}} |K_1(\phi(t)\tau, t; M)| \phi(t) d\tau \\ & \leq C e^{-\frac{n}{2}t} \int_0^1 \|\psi_1\|_{B_p^{s,q}} \phi(t)^{s-s'-2n\delta} \tau^{s-s'-2n\delta} |K_1(\phi(t)\tau, t; M)| \phi(t) d\tau \\ & \leq C_M e^{-\frac{n}{2}t} \|\psi_1\|_{B_p^{s,q}} e^{-Mt-(s-s'-2n\delta)t} (e^t - 1)^{1+s-s'-2n\delta} (e^t + 1)^{2M-1} \end{aligned}$$

that completes the proof of that case. Theorem is proved.  $\square$

### $B_p^{s,q} - B_p^{s',q}$ Estimates for Equation with Source

**Theorem 3.2** *Let  $\psi = \psi(x, t)$  be a solution of the Cauchy problem*

$$\psi_{tt} + n\psi_t - e^{-2t} A(x, \partial_x) \psi \pm m^2 \psi = f, \quad \psi(x, 0) = 0, \quad \psi_t(x, 0) = 0. \quad (3.2)$$

Define  $M = \sqrt{\frac{n^2}{4} - m^2}$  and  $m < n/2$  for the case of “plus”, and  $M = \sqrt{\frac{n^2}{4} + m^2}$  for the case of “minus”. Then  $\psi = \psi(x, t)$  satisfies the following  $B_p^{s,q} - B_p^{s',q}$  estimate:

$$\begin{aligned} \|\psi(x, t)\|_{B_p^{s',q}} & \leq C_M e^{-Mt} e^{-\frac{n}{2}t} e^{-t(s-s'-2n\delta)} \\ & \quad \times \int_0^t e^{\frac{n}{2}b} e^{Mb} (e^{t-b} - 1)^{1+s-s'-2n\delta} (e^{t-b} + 1)^{2M-1} \|f(x, b)\|_{B_p^{s,q}} db, \end{aligned} \quad (3.3)$$

for all  $t > 0$ , provided that  $s, s' \geq 0$ ,  $s, s' \geq 0$ ,  $(n+1)\delta \leq s - s'$ ,  $1 < p \leq 2$ ,  $\frac{1}{p} + \frac{1}{p'} = 1$ .

*Proof.* From (1.9) we have

$$\begin{aligned}\psi(x, t) &= 2e^{-\frac{n}{2}t} \int_0^t db \int_0^{e^{-b}-e^{-t}} dr e^{\frac{n}{2}b} v_f(x, r; b) 4^{-M} e^{M(b+t)} \left( (e^{-t} + e^{-b})^2 - r^2 \right)^{-\frac{1}{2}+M} \\ &\quad \times F\left(\frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(e^{-b} - e^{-t})^2 - r^2}{(e^{-b} + e^{-t})^2 - r^2}\right),\end{aligned}$$

where according to (2.3) of Theorem 2.1 we can write

$$\|v_f(x, r; b)\|_{B_{p'}^{s', q}} \leq C r^{s-s'-2n\delta} \|f(x, b)\|_{B_p^{s, q}} \quad \text{for all } r > 0.$$

Hence,

$$\begin{aligned}& \|\psi(x, t)\|_{B_{p'}^{s', q}} \\ & \leq 2e^{-\frac{n}{2}t} \int_0^t db \int_0^{e^{-b}-e^{-t}} dr e^{\frac{n}{2}b} \|v_f(x, r; b)\|_{B_{p'}^{s', q}} 4^{-M} e^{M(b+t)} \left( (e^{-t} + e^{-b})^2 - r^2 \right)^{-\frac{1}{2}+M} \\ & \quad \times \left| F\left(\frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(e^{-b} - e^{-t})^2 - r^2}{(e^{-b} + e^{-t})^2 - r^2}\right) \right| \\ & \leq C_M e^{Mt} e^{-\frac{n}{2}t} \int_0^t e^{\frac{n}{2}b} e^{Mb} \|f(x, b)\|_{B_p^{s, q}} db \int_0^{e^{-b}-e^{-t}} r^{s-s'-2n\delta} \left( (e^{-t} + e^{-b})^2 - r^2 \right)^{-\frac{1}{2}+M} \\ & \quad \times \left| F\left(\frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(e^{-b} - e^{-t})^2 - r^2}{(e^{-b} + e^{-t})^2 - r^2}\right) \right| dr.\end{aligned}$$

We set  $r = ye^{-t}$  and obtain

$$\begin{aligned}& \|\psi(x, t)\|_{B_{p'}^{s', q}} \\ & \leq C_M e^{-Mt} e^{-\frac{n}{2}t} e^{-t(s-s'-2n\delta)} \int_0^t e^{\frac{n}{2}b} e^{Mb} \|f(x, b)\|_{B_p^{s, q}} db \\ & \quad \times \int_0^{e^{t-b}-1} y^{s-s'-2n\delta} \left( (e^{t-b} + 1)^2 - y^2 \right)^{-\frac{1}{2}+M} \left| F\left(\frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(e^{t-b} - 1)^2 - y^2}{(e^{t-b} + 1)^2 - y^2}\right) \right| dy.\end{aligned}$$

Hence, we have to estimate the integral

$$\int_0^{z-1} y^a \left( (z+1)^2 - y^2 \right)^{-\frac{1}{2}+M} \left| F\left(\frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(z-1)^2 - y^2}{(z+1)^2 - y^2}\right) \right| dy,$$

where  $z = e^{t-b} > 1$  and  $a = s - s' - 2n\delta > -1$ . On the other hand, since  $M > 0$ , we have

$$\begin{aligned}& \int_0^{z-1} y^a \left( (z+1)^2 - y^2 \right)^{-\frac{1}{2}+M} \left| F\left(\frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(z-1)^2 - y^2}{(z+1)^2 - y^2}\right) \right| dy \\ & \leq C_M \int_0^{z-1} y^a \left( (z+1)^2 - y^2 \right)^{-\frac{1}{2}+M} dy \\ & = C_M \frac{1}{1+a} (e^{t-b} - 1)^{1+a} (e^{t-b} + 1)^{-1+2M} F\left(\frac{1+a}{2}, \frac{1}{2} - M; \frac{3+a}{2}; \frac{(e^{t-b} - 1)^2}{(e^{t-b} + 1)^2}\right) \\ & \leq C_M \frac{1}{1+a} (e^{t-b} - 1)^{1+a} (e^{t-b} + 1)^{-1+2M},\end{aligned}$$

since

$$\left| F\left(\frac{1+a}{2}, \frac{1}{2} - M; \frac{3+a}{2}; \frac{(e^{t-b} - 1)^2}{(e^{t-b} + 1)^2}\right) \right| \leq C_M.$$

This completes the proof of the estimate (3.3). Theorem is proved.  $\square$

The following corollaries can be easily verified.

**Corollary 3.3** *Let  $\psi = \psi(x, t)$  be a solution of the Cauchy problem considered in Theorem 3.2. Then for  $n \geq 2$  and  $M > 0$  one has the following estimate*

$$\|\psi(x, t)\|_{B_{p'}^{s', q}} \leq C_M e^{-(\frac{n}{2}-M)t} \int_0^t e^{(\frac{n}{2}-M)b} e^{-b(s-s'-2n\delta)} \|f(x, b)\|_{B_p^{s, q}} db,$$

provided that  $s, s' \geq 0$ ,  $(n+1)\delta \leq s - s'$ ,  $1 < p \leq 2$ ,  $\frac{1}{p} + \frac{1}{p'} = 1$ ,  $1 < p \leq 2$ ,  $s - s' - 2n\delta > -1$ , and  $\delta = 1/p - 1/2$ .

**Corollary 3.4** *Let  $\psi = \psi(x, t)$  be a solution of the Cauchy problem considered in Theorem 3.2. Then for  $n \geq 2$  and  $M > 0$  one has the following estimate*

$$\|\psi(x, t)\|_{H_{(s)}(\mathbb{R}^n)} \leq C_M e^{-(\frac{n}{2}-M)t} \int_0^t e^{(\frac{n}{2}-M)b} \|f(x, b)\|_{H_{(s)}(\mathbb{R}^n)} db,$$

## 4 Critical Mass $m^2 = (n^2 - 1)/4$

The value  $m^2 = (n^2 - 1)/4$  of the mass  $m$ , when  $M = 1/2$ , dramatically simplifies the function  $E(x, t; x_0, t_0; M)$ , and consequently,  $K_0(z, t; M)$  and  $K_1(z, t; M)$ . Indeed, in that case we have

$$E\left(x, t; x_0, t_0; \frac{1}{2}\right) = \frac{1}{2} e^{\frac{1}{2}(t_0+t)}, \quad E\left(z, t; 0, b; \frac{1}{2}\right) = \frac{1}{2} e^{\frac{1}{2}(b+t)},$$

while

$$K_0\left(z, t; \frac{1}{2}\right) = -\frac{1}{4} e^{\frac{1}{2}t}, \quad K_1\left(z, t; \frac{1}{2}\right) = \frac{1}{2} e^{\frac{1}{2}t}.$$

### 4.1 Integral Transform and Huygens' Principle

In [31] for the equation with  $A(x, \partial_x) = \Delta$  is proved that for *critical value*  $(n^2 - 1)/4$  of mass is the only value which makes equation Huygensian, that is, solutions of the equation obey Huygens' principle. For the equations with  $x$ -dependent coefficient we have the following theorem.

**Theorem 4.1** *Assume that  $m = \sqrt{n^2 - 1}/2$ . Then the solutions of the equation*

$$\psi_{tt} + n\psi_t - e^{-2t}A(x, \partial_x)\psi + m^2\psi = 0, \tag{4.1}$$

*obey the strong Huygens' Principle, whenever the solutions of the equation*

$$u_{tt} - A(x, \partial_x)u = 0,$$

*obey it.*

*Proof.* For the solution (1.9) of the equation (1.8) with the source term it follows

$$\psi(x, t) = e^{-\frac{n-1}{2}t} \int_0^t e^{\frac{n+1}{2}b} db \int_0^{e^{-b}-e^{-t}} v(x, r; b) dr,$$

where the function  $v(x, r; b)$  is defined by (1.5). In fact, if we denote by  $V_f(x, t; b)$  the solution of the problem

$$V_{tt} - A(x, \partial_x)V = 0, \quad V(x, 0) = 0, \quad V_t(x, 0) = f(x, b),$$

then

$$v(x, t; b) = \frac{\partial}{\partial t} V_f(x, t; b).$$

Hence,

$$\psi(x, t) = e^{-\frac{n-1}{2}t} \int_0^t e^{\frac{n+1}{2}b} V_f(x, e^{-b} - e^{-t}; b) db. \quad (4.2)$$

Further, due to (1.11), for the solution  $\psi$  of (1.10) we have

$$\begin{aligned} \psi(x, t) &= e^{-\frac{n-1}{2}t} v_{\psi_0}(x, 1 - e^{-t}) + \frac{n-1}{2} e^{-\frac{n-1}{2}t} \int_0^{1-e^{-t}} v_{\psi_0}(x, s) ds \\ &\quad + e^{-\frac{n-1}{2}t} \int_0^{1-e^{-t}} v_{\psi_1}(x, s) ds, \quad x \in \mathbb{R}^n, \quad t > 0, \end{aligned}$$

where the functions  $v_{\psi_0}$  and  $v_{\psi_1}$  are defined by (1.6). Now, if we denote by  $V_\varphi$  the solution of the problem

$$V_{tt} - A(x, \partial_x)V = 0, \quad V(x, 0) = 0, \quad V_t(x, 0) = \varphi(x),$$

then

$$v_\varphi(x, t) = \frac{\partial}{\partial t} V_\varphi(x, t),$$

and

$$\begin{aligned} \psi(x, t) &= e^{-\frac{n-1}{2}t} v_{\psi_0}(x, 1 - e^{-t}) + \frac{n-1}{2} e^{-\frac{n-1}{2}t} V_{\psi_0}(x, 1 - e^{-t}) \\ &\quad + e^{-\frac{n-1}{2}t} V_{\psi_1}(x, 1 - e^{-t}), \quad x \in \mathbb{R}^n, \quad t > 0, \end{aligned}$$

or, equivalently,

$$\begin{aligned} \psi(x, t) &= e^{-\frac{n-1}{2}t} \left( \frac{\partial V_{\psi_0}}{\partial t} \right) (x, 1 - e^{-t}) + \frac{n-1}{2} e^{-\frac{n-1}{2}t} V_{\psi_0}(x, 1 - e^{-t}) \\ &\quad + e^{-\frac{n-1}{2}t} V_{\psi_1}(x, 1 - e^{-t}), \quad x \in \mathbb{R}^n, \quad t > 0. \end{aligned}$$

Theorem is proved. □

## 4.2 The Critical Case. Asymptotic Expansions of Solutions at Infinite Time

In this subsection we generalize the results of [31], about the asymptotic expansions of solutions at the large time, to the equations with  $x$ -dependent coefficients. For  $\varphi \in C_0^\infty(\mathbb{R}^n)$  let  $v_\varphi(x, t)$  be a solution of the Cauchy problem

$$u_{tt} - A(x, \partial_x)u = 0, \quad u(x, 0) = 0, \quad u_t(x, 0) = \varphi(x). \quad (4.3)$$

Denote

$$v_\varphi(x) := v_\varphi(x, 1), \quad V_\varphi(x) := V_\varphi(x, 1).$$

In order to write the complete asymptotic expansion of the solutions, we define the functions

$$V_\varphi^{(k)}(x) := \frac{(-1)^k}{k!} \left[ \left( \frac{\partial}{\partial t} \right)^k V_\varphi(x, t) \right]_{t=1} \in C_0^\infty(\mathbb{R}^n), \quad k = 1, 2, \dots$$

Then, for every integer  $N \geq 1$  we have

$$V_\varphi(x, 1 - e^{-t}) = \sum_{k=0}^{N-1} V_\varphi^{(k)}(x) e^{-kt} + R_{V_\varphi, N}(x, t), \quad R_{V_\varphi, N} \in C^\infty,$$

where with the constant  $C(\varphi)$  the remainder  $R_{V_\varphi, N}$  satisfies the inequality

$$|R_{V_\varphi, N}(x, t)| \leq C(\varphi) e^{-Nt} \quad \text{for all } x \in \mathbb{R}^n \text{ and all } t \in [0, \infty).$$

Moreover, the support of the remainder  $R_{V_\varphi, N}$  is in the cylinder

$$\text{supp } R_{V_\varphi, N} \subseteq \{x \in \mathbb{R}^n; \text{dist}(x, \text{supp } \varphi) \leq c_0\} \times [0, \infty).$$

where  $c_0$  is speed of propagation of the equation (4.3). Analogously, we define

$$v_\varphi^{(k)}(x) = \frac{(-1)^k}{k!} \left[ \left( \frac{\partial}{\partial t} \right)^k v_\varphi(x, t) \right]_{t=1} \in C_0^\infty(\mathbb{R}^n), \quad k = 1, 2, \dots,$$

and the remainder  $R_{v_\varphi, N}$

$$v_\varphi(x, 1 - e^{-t}) = \sum_{k=0}^{N-1} v_\varphi^{(k)}(x) e^{-kt} + R_{v_\varphi, N}(x, t), \quad R_{v_\varphi, N} \in C^\infty,$$

such that

$$|R_{v_\varphi, N}(x, t)| \leq C(\varphi) e^{-Nt} \quad \text{for all } x \in \mathbb{R}^n \text{ and all } t \in [0, \infty).$$

Further, we introduce the polynomial in  $z$  with the smooth in  $x \in \mathbb{R}^n$  coefficients as follows:

$$\psi_{asymp}^{(N)}(x, z) = z^{\frac{n-1}{2}} \left( \sum_{k=0}^{N-1} v_{\varphi_0}^{(k)}(x) z^k + \frac{n-1}{2} \sum_{k=0}^{N-1} V_{\varphi_0}^{(k)}(x) z^k \right) + z^{\frac{n-1}{2}} \sum_{k=0}^{N-1} V_{\varphi_1}^{(k)}(x) z^k,$$

where  $x \in \mathbb{R}^n$ ,  $z \in \mathbb{C}$ . This allows us to prove the following asymptotic expansion

$$\psi(x, t) = \psi_{asymp}^{(N)}(x, e^{-t}) + O(e^{-Nt - \frac{n-1}{2}t})$$

for large  $t$  uniformly for  $x \in \mathbb{R}^n$ . Thus, we have proved the next theorem.

**Theorem 4.2** Suppose that  $m = \sqrt{n^2 - 1}/2$ . Then, for every integer positive  $N$  the solution of the equation (4.1) with the initial values  $\varphi_0, \varphi_1 \in C_0^\infty(\mathbb{R}^n)$  has the following asymptotic expansion at infinity:

$$\psi(x, t) \sim \psi_{\text{asympt}}^{(N)}(x, e^{-t}),$$

in the sense that for every integer positive  $N$  the following estimate is valid:

$$\|\psi(x, t) - \psi_{\text{asympt}}^{(N)}(x, e^{-t})\|_{L^\infty(\mathbb{R}^n)} \leq C(\varphi_0, \varphi_1) e^{-Nt - \frac{n-1}{2}t} \quad \text{for large } t.$$

**Remark 4.3** If we take into account the relation  $v_\varphi(x, t) = \frac{\partial}{\partial t} V_\varphi(x, t)$ , then

$$v_\varphi^{(k)}(x) = -(k+1)V_\varphi^{(k+1)}(x)$$

and, consequently, the function  $\Phi_{\text{asympt}}^{(N)}(x, z)$  can be rewritten as follows:

$$\begin{aligned} & \psi_{\text{asympt}}^{(N)}(x, z) \\ &= z^{\frac{n-1}{2}} \left( \sum_{k=0}^{N-1} (-1)(k+1)V_{\varphi_0}^{(k+1)}(x)z^k + \frac{n-1}{2} \sum_{k=0}^{N-1} V_{\varphi_0}^{(k)}(x)z^k \right) + z^{\frac{n-1}{2}} \sum_{k=0}^{N-1} V_{\varphi_1}^{(k)}(x)z^k \\ &= z^{\frac{n-1}{2}} \sum_{k=0}^{N-1} \left( \frac{n-1}{2} V_{\varphi_0}^{(k)}(x) - (k+1)V_{\varphi_0}^{(k+1)}(x) + V_{\varphi_1}^{(k)}(x) \right) z^k. \end{aligned}$$

### 4.3 The Critical Case. $B_p^{s,q} - B_{p'}^{s',q}$ -Estimates

**Lemma 4.4** Suppose that  $m = \sqrt{n^2 - 1}/2$ ,  $(n+1)\delta \leq 1+s-s'$ ,  $s, s' \geq 0$ ,  $1 \leq p \leq 2$ ,  $1/p+1/p' = 1$ ,  $s-s'-2n\delta > -2$ , and  $\delta = 1/p - 1/2$ . If  $\psi_0 = \psi_1 = 0$ , then for the solution  $\psi = \psi(x, t)$  of the equation (1.8), (0.3) the following estimate holds

$$\|\psi(x, t)\|_{B_{p'}^{s',q}} \leq C e^{-\frac{n-1}{2}t} \int_0^t e^{\frac{n+1}{2}b} (e^{-b} - e^{-t})^{1+s-s'-2n\delta} \|f(x, b)\|_{B_p^{s,q}} db, \quad t > 0.$$

For the solution  $\psi = \psi(x, t)$  of the Cauchy problem (1.10) if  $\psi_0 = 0$ , and  $s-s'-2n\delta > -1$ ,  $(n+1)\delta \leq 1+s-s'$ , then

$$\|\psi(x, t)\|_{B_{p'}^{s',q}} \leq C e^{-\frac{n-1}{2}t} (1 - e^{-t})^{1+s-s'-2n\delta} \|\psi_1\|_{B_p^{s,q}}, \quad t > 0.$$

If  $\psi_1 = 0$  and  $(n+1)\delta \leq s-s'$ , then

$$\|\psi(x, t)\|_{B_{p'}^{s',q}} \leq C e^{-\frac{n-1}{2}t} (1 - e^{-t})^{s-s'-2n\delta} \|\psi_0\|_{B_p^{s,q}}, \quad t > 0.$$

*Proof.* According to (4.2) and Theorem 2.1, if  $\psi_0 = \psi_1 = 0$  and  $(n+1)\delta \leq 1+s-s'$ , then

$$\begin{aligned} \|\psi(x, t)\|_{B_{p'}^{s',q}} &\leq \|e^{-\frac{n-1}{2}t} \int_0^t e^{\frac{n+1}{2}b} V_f(x, e^{-b} - e^{-t}; b) db\|_{B_{p'}^{s',q}} \\ &\leq e^{-\frac{n-1}{2}t} \int_0^t e^{\frac{n+1}{2}b} \|V_f(x, e^{-b} - e^{-t}; b)\|_{B_{p'}^{s',q}} db \\ &\leq C e^{-\frac{n-1}{2}t} \int_0^t e^{\frac{n+1}{2}b} (e^{-b} - e^{-t})^{1+s-s'-2n\delta} \|f(x, b)\|_{B_p^{s,q}} db, \quad t > 0. \end{aligned}$$



In particular,

$$\begin{aligned}
& \|\psi(x, t)\|_{B_{p'}^{s', q}} \\
& \leq C \left( \sup_{0 \leq b \leq t} \|f(x, b)\|_{B_p^{s, q}} \right) e^{-\frac{n-1}{2}t} \int_0^t e^{\frac{n+1}{2}b} (e^{-b} - e^{-t})^{1+s-s'-2n\delta} db \\
& \leq C \left( \sup_{0 \leq b \leq t} \|f(x, b)\|_{B_p^{s, q}} \right) e^{-\frac{n-1}{2}t} e^{-t(1+s-s'-2n\delta)} \int_0^t e^{\frac{n+1}{2}b} (e^{t-b} - 1)^{1+s-s'-2n\delta} db, \quad t > 0.
\end{aligned}$$

For the case  $s = s'$ ,  $\delta = 0$ , and  $p = p' = 2$  we obtain the estimate

$$\begin{aligned}
\|\psi(x, t)\|_{H_{(s)}(\mathbb{R}^n)} & \leq C \left( \sup_{0 \leq b \leq t} \|f(x, b)\|_{H_{(s)}(\mathbb{R}^n)} \right) e^{-\frac{n-1}{2}t} \int_0^t e^{\frac{n-1}{2}b} db \\
& \leq C \left( \sup_{0 \leq b \leq t} \|f(x, b)\|_{H_{(s)}(\mathbb{R}^n)} \right), \quad t > 0.
\end{aligned}$$

Further, if  $f \equiv 0$ ,  $\psi_0 = 0$ , and  $(n+1)\delta \leq 1 + s - s'$ , then

$$\|\psi(x, t)\|_{B_{p'}^{s', q}} \leq C e^{-\frac{n-1}{2}t} (1 - e^{-t})^{1+s-s'-2n\delta} \|\psi_1\|_{B_p^{s, q}}, \quad t > 0,$$

If  $f \equiv 0$ ,  $\psi_1 = 0$ , and  $(n+1)\delta \leq s - s'$ , then

$$\|\psi(x, t)\|_{B_{p'}^{s', q}} \leq C e^{-\frac{n-1}{2}t} (1 - e^{-t})^{s-s'-2n\delta} \|\psi_0\|_{B_p^{s, q}}, \quad t > 0.$$

The lemma is proved. □

## 5 Global Existence. Small Data Solutions

Let  $\psi = \psi(x, t)$  be a solution of the Cauchy problem (3.2) with either  $M = \sqrt{\frac{n^2}{4} - m^2}$  and  $m \in (0, n/2)$  for the case of “plus”, or  $M = \sqrt{\frac{n^2}{4} + m^2}$  for the case of “minus”. Then for  $n \geq 2$ , according to Corollary 3.3 one has the following estimate

$$\|\psi(x, t)\|_{B_2^{s, q}} \leq C_M e^{-(\frac{n}{2}-M)t} \int_0^t e^{(\frac{n}{2}-M)b} \|f(x, b)\|_{B_2^{s, q}} db,$$

For the equation with “plus” and large mass,  $m \geq n/2$ , and with the curved mass  $M = \sqrt{m^2 - n^2/4}$ , according to (2.12) one has the following estimate

$$\|\psi(x, t)\|_{B_2^{s, q}} \leq C_M e^{-\frac{n}{2}t} \int_0^t e^{\frac{n}{2}b} (1 + t - b)^{1-\text{sgn}M} \|f(x, b)\|_{B_2^{s, q}} db.$$

Here the rate of exponential factors is independent of the curved mass  $M$  and, consequently, of the mass  $m$ . These statements follow immediately from Corollary 3.3 and Section 2.3.

The last estimates and the fixed point theorem allow us to prove global existence in the Cauchy problem for the semilinear equation

$$\psi_{tt} + n\psi_t - e^{-2t}A(x, \partial_x)\psi + m^2\psi = F(\psi).$$

We study the Cauchy problem (0.2), (0.3) through the integral equation. To determine that integral equation we appeal to the operator

$$G := \mathcal{K} \circ \mathcal{E}\mathcal{E}_{\mathcal{A}},$$

where

$$\mathcal{E}\mathcal{E}_{\mathcal{A}}[f](x, t; b) = v(x, t; b)$$

and the function  $v(x, t; b)$  is a solution to the Cauchy problem for the equation (1.6), while  $\mathcal{K}$  is introduced either by (1.9),

$$\begin{aligned} \mathcal{K}[v](x, t) &:= 2e^{-\frac{n}{2}t} \int_0^t db \int_0^{e^{-b}-e^{-t}} dr e^{\frac{n}{2}b} v(x, r; b) E(r, t; 0, b; -iM) \\ &= 2e^{-\frac{n}{2}t} \int_0^t db \int_0^{e^{-b}-e^{-t}} dr e^{\frac{n}{2}b} \mathcal{E}\mathcal{E}_{\mathcal{A}}[f](x, r; b) E(r, t; 0, b; -iM), \end{aligned} \quad (5.1)$$

for the large mass  $m$ , or by

$$\begin{aligned} \mathcal{K}[v](x, t) &:= 2e^{-\frac{n}{2}t} \int_0^t db \int_0^{e^{-b}-e^{-t}} dr e^{\frac{n}{2}b} v(x, r; b) E(r, t; 0, b; M) \\ &= 2e^{-\frac{n}{2}t} \int_0^t db \int_0^{e^{-b}-e^{-t}} dr e^{\frac{n}{2}b} \mathcal{E}\mathcal{E}_{\mathcal{A}}[f](x, r; b) E(r, t; 0, b; M), \end{aligned} \quad (5.2)$$

for the small mass  $m$ . Thus, the Cauchy problem (0.2), (0.3) leads to the following integral equation

$$\psi(x, t) = \psi_0(x, t) + G[F(\psi)](x, t). \quad (5.3)$$

Every solution to the equation (0.2) solves also the last integral equation with some function  $\psi_0(x, t)$ , which, in fact, is a solution of the Cauchy problem (2.11).

## 5.1 Solvability of the Integral Equation associated with Klein-Gordon Equation

Consider the integral equation (5.3) where  $\psi_0 = \psi_0(x, t)$  is a given function. Every solution to the equation (0.2) solves also the last integral equation with some function  $\psi_0 = \psi_0(x, t)$ . In order to solve the integral equation (5.3), we apply the Banach fixed-point theorem. To estimate nonlinear term we use the Lipschitz condition ( $\mathcal{L}$ ). Evidently, the condition ( $\mathcal{L}$ ) imposes some restrictions on  $n$ ,  $\alpha$ ,  $s$ . Now we consider the integral equation (5.3), where the function  $\psi_0 \in C([0, \infty); B_p^{s,q})$  is given. We note here that any classical solution to the equation (0.2) solves also the integral equation (5.3) with some function  $\psi_0(t, x)$ , which is classical solution to the Cauchy problem for the linear equation (1.10).

Solvability of the integral equation (5.3) is determined by the operator  $G = \mathcal{K} \circ \mathcal{E}\mathcal{E}_{\mathcal{A}}$ . We start with the case of Sobolev space  $H_{(s)}(\mathbb{R}^n)$  with  $s > n/2$ , which is an algebra. In the next theorem operator  $\mathcal{K}$  is generated by the linear part of the equation (0.2).

**Theorem 5.1** *Assume that  $F$  is Lipschitz continuous with exponent  $\alpha > 0$  in the space  $H_{(s)}(\mathbb{R}^n)$ ,  $s > n/2$ , and  $F(x, 0) = 0$  for all  $x \in \mathbb{R}^n$ . Then, there exists sufficiently small  $\varepsilon_0 > 0$  such that, for every given function  $\psi_0(x, t) \in X(\varepsilon, s, \gamma_0)$ ,  $\varepsilon < \varepsilon_0$ , such that*

$$\begin{aligned} \sup_{t \in [0, \infty)} e^{\gamma_0 t} \|\psi_0(\cdot, t)\|_{H_{(s)}(\mathbb{R}^n)} &< \varepsilon, \\ \gamma_0 &\leq \frac{n}{2} - \sqrt{\frac{n^2}{4} - m^2} \quad \text{if} \quad 0 < m < \frac{n}{2}, \quad \text{while} \\ 0 &\leq \gamma_0 \leq \frac{n}{2} \quad \text{if} \quad \frac{n}{2} \leq m, \end{aligned}$$

the integral equation (5.3) has a unique solution  $\psi(x, t) \in X(2\varepsilon, s, \gamma)$  with

$$\begin{cases} 0 < \gamma < \frac{1}{\alpha+1}\gamma_0 & \text{if } 0 < m < \frac{n}{2}, \\ \gamma \leq \min\left\{\gamma_0, \frac{n}{2(\alpha+1)}\right\} & \text{if } \frac{n}{2} \leq m. \end{cases}$$

Thus, for the solution  $\psi$  the following estimate holds:

$$\sup_{t \in [0, \infty)} e^{\gamma t} \|\psi(\cdot, t)\|_{H_{(s)}(\mathbb{R}^n)} < 2\varepsilon.$$

*Proof.* Consider the mapping

$$S[\psi](x, t) := \psi_0(x, t) + G[F(\psi)](x, t). \quad (5.4)$$

We are going to prove that  $S$  maps  $X(R, s, \gamma)$  into itself and is a contraction provided that  $\varepsilon$  and  $R$  are sufficiently small.

**The case of small physical mass,  $m \in (0, \frac{n}{2})$ .** In this case the operator  $\mathcal{K}$  is given by (5.2) and  $M = \sqrt{\frac{n^2}{4} - m^2} > 0$ . Corollary 3.4 with  $s > n/2$ ,  $\gamma = \frac{1}{\alpha+1}(\frac{n}{2} - M - \delta) > 0$  and  $\delta > 0$  implies

$$\begin{aligned} \|S[\psi](\cdot, t)\|_{H_{(s)}(\mathbb{R}^n)} &\leq \|\psi_0(\cdot, t)\|_{H_{(s)}(\mathbb{R}^n)} + \|G[F(\cdot, \psi)](\cdot, t)\|_{H_{(s)}(\mathbb{R}^n)} \\ &\leq \|\psi_0(\cdot, t)\|_{H_{(s)}(\mathbb{R}^n)} + C_M e^{-(\frac{n}{2}-M)t} \int_0^t e^{(\frac{n}{2}-M)b} \|F(\cdot, \psi)(\cdot, b)\|_{H_{(s)}(\mathbb{R}^n)} db \\ &\leq \|\psi_0(\cdot, t)\|_{H_{(s)}(\mathbb{R}^n)} + C_M e^{-\gamma(\alpha+1)t-\delta t} \int_0^t e^{\gamma(\alpha+1)b+\delta b} \|F(\cdot, \psi)(\cdot, b)\|_{H_{(s)}(\mathbb{R}^n)} db. \end{aligned}$$

Taking into account the Condition  $(\mathcal{L})$  we arrive at

$$\begin{aligned} \|S[\psi](\cdot, t)\|_{H_{(s)}(\mathbb{R}^n)} &\leq \|\psi_0(\cdot, t)\|_{H_{(s)}(\mathbb{R}^n)} + C_M e^{-\gamma(\alpha+1)t-\delta t} \int_0^t e^{\gamma(\alpha+1)b+\delta b} \|\psi(\cdot, b)\|_{H_{(s)}(\mathbb{R}^n)}^{\alpha+1} db \\ &\leq \|\psi_0(\cdot, t)\|_{H_{(s)}(\mathbb{R}^n)} + C_M e^{-\gamma(\alpha+1)t-\delta t} \int_0^t e^{\delta b} \left( e^{\gamma b} \|\psi(\cdot, b)\|_{H_{(s)}(\mathbb{R}^n)} \right)^{\alpha+1} db. \end{aligned}$$

Then

$$\begin{aligned} e^{\gamma t} \|S[\psi](x, t)\|_{H_{(s)}(\mathbb{R}^n)} &\leq e^{\gamma(\alpha+1)t} \|S[\psi](x, t)\|_{H_{(s)}(\mathbb{R}^n)} \\ &\leq e^{\gamma(\alpha+1)t} \|\psi_0(\cdot, t)\|_{H_{(s)}(\mathbb{R}^n)} + C_M \left( \sup_{\tau \in [0, \infty)} e^{\gamma \tau} \|\psi(\cdot, \tau)\|_{H_{(s)}(\mathbb{R}^n)} \right)^{\alpha+1} e^{-\delta t} \int_0^t e^{\delta b} db \\ &\leq e^{\gamma_0 t} \|\psi_0(\cdot, t)\|_{H_{(s)}(\mathbb{R}^n)} + C_M \delta^{-1} \left( \sup_{\tau \in [0, \infty)} e^{\gamma \tau} \|\psi(\cdot, \tau)\|_{H_{(s)}(\mathbb{R}^n)} \right)^{\alpha+1}, \end{aligned}$$

and

$$\begin{aligned} &\sup_{t \in [0, \infty)} e^{\gamma t} \|S[\psi](x, t)\|_{H_{(s)}(\mathbb{R}^n)} \\ &\leq \sup_{t \in [0, \infty)} e^{\gamma_0 t} \|\psi_0(\cdot, t)\|_{H_{(s)}(\mathbb{R}^n)} + C_M \delta^{-1} \left( \sup_{t \in [0, \infty)} e^{\gamma t} \|\psi(\cdot, t)\|_{H_{(s)}(\mathbb{R}^n)} \right)^{\alpha+1}. \end{aligned}$$

In particular, if  $\gamma_0 = \frac{n}{2} - M > 0$ , then, with  $\delta > 0$  such that  $\gamma(\alpha + 1) = \frac{n}{2} - M - \delta < \gamma_0$ , we have

$$\begin{aligned} & \sup_{t \in [0, \infty)} e^{\gamma t} \|S[\psi](\cdot, t)\|_{H_{(s)}(\mathbb{R}^n)} \\ & \leq \sup_{t \in [0, \infty)} e^{(\frac{n}{2} - M)t} \|\psi_0(\cdot, t)\|_{H_{(s)}(\mathbb{R}^n)} + C \left( \sup_{t \in [0, \infty)} e^{\gamma t} \|\psi(\cdot, t)\|_{H_{(s)}(\mathbb{R}^n)} \right)^{\alpha+1}. \end{aligned}$$

Thus, the last inequality proves that the operator  $S$  maps  $X(R, s, \gamma)$  into itself if  $\varepsilon$  and  $R$  are sufficiently small, namely, if  $\varepsilon + CR^{\alpha+1} < R$ .

It remains to prove that  $S$  is a contraction mapping. As a matter of fact, we just need to apply estimate (0.1) and get the contraction property from

$$e^{\gamma t} \|S[\psi](\cdot, t) - S[\tilde{\psi}](\cdot, t)\|_{H_{(s)}(\mathbb{R}^n)} \leq CR(t)^\alpha d(\psi, \tilde{\psi}),$$

where  $R(t) := \max\left\{ \sup_{0 \leq \tau \leq t} e^{\gamma \tau} \|\psi(\cdot, \tau)\|_{H_{(s)}(\mathbb{R}^n)}, \sup_{0 \leq \tau \leq t} e^{\gamma \tau} \|\tilde{\psi}(\cdot, \tau)\|_{H_{(s)}(\mathbb{R}^n)} \right\} \leq R$ . Indeed, due to Condition  $(\mathcal{L})$ , we have

$$\begin{aligned} & \|S[\psi](\cdot, t) - S[\tilde{\psi}](\cdot, t)\|_{H_{(s)}(\mathbb{R}^n)} = \|G[(F(\cdot, \psi) - F(\cdot, \tilde{\psi}))](\cdot, t)\|_{H_{(s)}(\mathbb{R}^n)} \\ & \leq C_M e^{-(\frac{n}{2} - M)t} \int_0^t e^{(\frac{n}{2} - M)b} \|(F(\cdot, \psi) - F(\cdot, \tilde{\psi}))(\cdot, b)\|_{H_{(s)}(\mathbb{R}^n)} db \\ & \leq C_M e^{-\gamma(\alpha+1)t - \delta t} \int_0^t e^{\gamma(\alpha+1)b + \delta b} \|(F(\cdot, \psi) - F(\cdot, \tilde{\psi}))(\cdot, b)\|_{H_{(s)}(\mathbb{R}^n)} db \\ & \leq C_M e^{-\gamma(\alpha+1)t - \delta t} \int_0^t e^{\gamma(\alpha+1)b + \delta b} \|\psi(\cdot, b) - \tilde{\psi}(\cdot, b)\|_{H_{(s)}(\mathbb{R}^n)} \\ & \quad \times \left( \|\psi(\cdot, b)\|_{H_{(s)}(\mathbb{R}^n)}^\alpha + \|\tilde{\psi}(\cdot, b)\|_{H_{(s)}(\mathbb{R}^n)}^\alpha \right) db. \end{aligned}$$

Thus, taking into account the last estimate and the definition of the metric  $d(\psi, \tilde{\psi})$ , we obtain

$$\begin{aligned} & e^{\gamma(\alpha+1)t} \|S[\psi](\cdot, t) - S[\tilde{\psi}](\cdot, t)\|_{H_{(s)}(\mathbb{R}^n)} \\ & \leq C_M e^{-\delta t} \int_0^t e^{\gamma(\alpha+1)b + \delta b} \|\psi(\cdot, b) - \tilde{\psi}(\cdot, b)\|_{H_{(s)}(\mathbb{R}^n)} \left( \|\psi(\cdot, b)\|_{H_{(s)}(\mathbb{R}^n)}^\alpha + \|\tilde{\psi}(\cdot, b)\|_{H_{(s)}(\mathbb{R}^n)}^\alpha \right) db \\ & \leq C_M e^{-\delta t} \int_0^t e^{\delta b} \left( \max_{0 \leq \tau \leq b} e^{\gamma \tau} \|\psi(\cdot, \tau) - \tilde{\psi}(\cdot, \tau)\|_{H_{(s)}(\mathbb{R}^n)} \right) \\ & \quad \times \left( \left( \max_{0 \leq \tau \leq b} e^{\gamma \tau} \|\psi(\cdot, \tau)\|_{H_{(s)}(\mathbb{R}^n)} \right)^\alpha + \left( \max_{0 \leq \tau \leq b} e^{\gamma \tau} \|\tilde{\psi}(\cdot, \tau)\|_{H_{(s)}(\mathbb{R}^n)} \right)^\alpha \right) db \\ & \leq C_{M, \alpha} d(\psi, \tilde{\psi}) R(t)^\alpha e^{-\delta t} \int_0^t e^{\delta b} db \\ & \leq C_{M, \alpha} \delta^{-1} d(\psi, \tilde{\psi}) R(t)^\alpha. \end{aligned}$$

Consequently,

$$e^{\gamma t} \|S[\psi](\cdot, t) - S[\tilde{\psi}](\cdot, t)\|_{H_{(s)}(\mathbb{R}^n)} \leq C_{M, \alpha} \delta^{-1} R(t)^\alpha d(\psi, \tilde{\psi}).$$

Set  $R := \sup_{t \in [0, \infty)} R(t)$ . Then we choose  $\varepsilon$  and  $R$  such that  $C_{M, \alpha} \delta^{-1} R^\alpha < 1$ . Banach's fixed point theorem completes the proof for the case of small physical mass.

**The case of the large physical mass**  $m \geq n/2$ . In this case the operator  $\mathcal{K}$  is given by (5.1). We set  $\gamma \leq \min\{\gamma_0, \frac{n}{2(\alpha+1)}\}$  in the definition of metric of the space  $X(R, s, \gamma)$ . Then, due to (2.13), we have

$$\begin{aligned} \|S[\psi](\cdot, t)\|_{H_{(s)}(\mathbb{R}^n)} &\leq \|\psi_0(\cdot, t)\|_{H_{(s)}(\mathbb{R}^n)} + \|G[F(\cdot, \psi)](\cdot, t)\|_{H_{(s)}(\mathbb{R}^n)} \\ &\leq \|\psi_0(\cdot, t)\|_{H_{(s)}(\mathbb{R}^n)} + C_M e^{-\frac{n}{2}t} \int_0^t e^{\frac{n}{2}b} (1+t-b)^{1-\text{sgn}M} \|F(\psi)(\cdot, b)\|_{H_{(s)}(\mathbb{R}^n)} db \\ &\leq \|\psi_0(\cdot, t)\|_{H_{(s)}(\mathbb{R}^n)} + C_{M,\alpha} e^{-\frac{n}{2}t} \int_0^t e^{\frac{n}{2}b} (1+t-b)^{1-\text{sgn}M} \|\psi(\cdot, b)\|_{H_{(s)}(\mathbb{R}^n)}^{\alpha+1} db. \end{aligned}$$

Hence

$$\begin{aligned} &e^{\gamma t} \|S[\psi](\cdot, t)\|_{H_{(s)}(\mathbb{R}^n)} \\ &\leq e^{\gamma t} \|\psi_0(\cdot, t)\|_{H_{(s)}(\mathbb{R}^n)} \\ &\quad + C_{M,\alpha} \left( \sup_{\tau \in [0, \infty)} e^{\gamma \tau} \|\psi(\cdot, \tau)\|_{H_{(s)}(\mathbb{R}^n)} \right)^{\alpha+1} e^{-(\frac{n}{2}-\gamma)t} \int_0^t e^{(\frac{n}{2}-\gamma(\alpha+1))b} (1+t-b)^{1-\text{sgn}M} db \\ &\leq e^{\gamma t} \|\psi_0(\cdot, t)\|_{H_{(s)}(\mathbb{R}^n)} + C_{M,\alpha} \left( \sup_{\tau \in [0, \infty)} e^{\gamma \tau} \|\psi(\cdot, \tau)\|_{H_{(s)}(\mathbb{R}^n)} \right)^{\alpha+1} e^{-\gamma \alpha t}. \end{aligned}$$

Then we choose  $\varepsilon$  and  $R$  such that  $\varepsilon + C_{M,\alpha} R^{\alpha+1} < R$ .

To prove that  $S$  is a contraction mapping, we just need to apply estimate (0.1) and get the contraction property from

$$e^{\gamma t} \|S[\psi](\cdot, t) - S[\tilde{\psi}](\cdot, t)\|_{H_{(s)}(\mathbb{R}^n)} \leq CR(t)^\alpha d(\psi, \tilde{\psi}),$$

where  $R(t) := \max\{\sup_{0 \leq \tau \leq t} e^{\gamma \tau} \|\psi(\cdot, \tau)\|_{H_{(s)}(\mathbb{R}^n)}, \sup_{0 \leq \tau \leq t} e^{\gamma \tau} \|\tilde{\psi}(\cdot, \tau)\|_{H_{(s)}(\mathbb{R}^n)}\} \leq R$ . Indeed, we have

$$\begin{aligned} &e^{\gamma t} \|S[\psi](\cdot, t) - S[\tilde{\psi}](\cdot, t)\|_{H_{(s)}(\mathbb{R}^n)} = e^{\gamma t} \|G[(F(\cdot, \psi) - F(\cdot, \tilde{\psi}))](\cdot, t)\|_{H_{(s)}(\mathbb{R}^n)} \\ &\leq C_M e^{-(\frac{n}{2}-\gamma)t} \int_0^t e^{\frac{n}{2}b} (1+t-b)^{1-\text{sgn}M} \|(F(\cdot, \psi) - F(\cdot, \tilde{\psi}))(\cdot, b)\|_{H_{(s)}(\mathbb{R}^n)} db \\ &\leq C_{M,\alpha} e^{-(\frac{n}{2}-\gamma)t} \int_0^t e^{\frac{n}{2}b} (1+t-b)^{1-\text{sgn}M} \|\psi(\cdot, b) - \tilde{\psi}(\cdot, b)\|_{H_{(s)}(\mathbb{R}^n)} \\ &\quad \times \left( \|\psi(\cdot, b)\|_{H_{(s)}(\mathbb{R}^n)}^\alpha + \|\tilde{\psi}(\cdot, b)\|_{H_{(s)}(\mathbb{R}^n)}^\alpha \right) db. \end{aligned}$$

Thus, taking into account the last estimate and a definition of the metric, we obtain

$$\begin{aligned} &e^{\gamma t} \|S[\psi](\cdot, t) - S[\tilde{\psi}](\cdot, t)\|_{H_{(s)}(\mathbb{R}^n)} \\ &\leq C_{M,\alpha} e^{-(\frac{n}{2}-\gamma)t} \int_0^t e^{(\frac{n}{2}-\gamma(\alpha+1))b} (1+t-b)^{1-\text{sgn}M} e^{\gamma b} \|\psi(\cdot, b) - \tilde{\psi}(\cdot, b)\|_{H_{(s)}(\mathbb{R}^n)} \\ &\quad \times \left( (e^{\gamma b} \|\psi(\cdot, b)\|_{H_{(s)}(\mathbb{R}^n)})^\alpha + (e^{\gamma b} \|\tilde{\psi}(\cdot, b)\|_{H_{(s)}(\mathbb{R}^n)})^\alpha \right) db \\ &\leq C_{M,\alpha} d(\psi, \tilde{\psi}) R(t)^\alpha e^{-(\frac{n}{2}-\gamma)t} \int_0^t e^{(\frac{n}{2}-\gamma(\alpha+1))b} (1+t-b)^{1-\text{sgn}M} db, \end{aligned}$$

and, consequently,

$$e^{\gamma t} \|S[\psi](\cdot, t) - S[\tilde{\psi}](\cdot, t)\|_{H_{(s)}(\mathbb{R}^n)} \leq C_{M,\alpha} d(\psi, \tilde{\psi}) R(t)^\alpha e^{-\gamma \alpha t}.$$

Set  $R := \sup_{t \in [0, \infty)} R(t)$ . Then we choose  $\varepsilon$  and  $R$  such that  $C_{M,\alpha} R^\alpha < 1$ . Banach's fixed point theorem completes the proof of theorem.  $\square$

## 5.2 Proof of Theorem 0.1

**The case of the small physical mass inside of Higuchi bound,**  $m < \sqrt{n^2 - 1}/2$ . In this case the operator  $\mathcal{K}$  is given by (5.2) and  $M = \sqrt{\frac{n^2}{4} - m^2}$ . Then for the function  $\psi_0 = \psi_0(x, t)$ , that is, for the solution of the Cauchy problem (2.11) and for  $s > \frac{n}{2}$ ,  $p = p' = 2$ ,  $n \geq 2$ , according to Theorem 3.1 we have the estimate

$$\|\psi_0(\cdot, t)\|_{H_{(s)}(\mathbb{R}^n)} \leq C_{M,n,p,q,s} e^{(M-\frac{n}{2})t} \left\{ \|\psi_0\|_{H_{(s)}(\mathbb{R}^n)} + \|\psi_1\|_{H_{(s)}(\mathbb{R}^n)} \right\}. \quad (5.5)$$

For every initial functions  $\psi_0(x)$  and  $\psi_1(x)$  the function  $\psi = \psi_0(x, t)$  belongs to the space  $X(R, s, \gamma)$ , where the operator  $S$  (5.4) is a contraction. In the case of  $n = 3$  that means  $m^2 < 2$ . Theorem 5.1 completes the proof of the existence of the global solution.

**The case of the critical mass,**  $m = \sqrt{n^2 - 1}/2$ . We apply Lemma 4.4 that shows that the estimate (5.5) holds with  $M = 1/2$ .

**The case of the small physical mass outside of Higuchi bound,**  $\sqrt{n^2 - 1}/2 < m < n/2$ . In this case  $\psi_0(x) = 0$  and again, the function  $\psi = \psi_0(x, t)$  belongs to the space  $X(R, s, \gamma)$ , where the operator  $S$  (5.4) is a contraction.

**The case of the large physical mass**  $m \geq n/2$ . In this case the operator  $\mathcal{K}$  is given by (5.1). Then for the function  $\psi_0$ , that is for the solution of the Cauchy problem (2.11) and for  $s > \frac{n}{2}$ ,  $p = p' = 2$ ,  $n \geq 2$  we apply the estimate (2.14),

$$\begin{aligned} e^{\gamma_0 t} \|\psi_0(\cdot, t)\|_{H_{(s)}(\mathbb{R}^n)} &\leq C_M e^{\gamma_0 t - \frac{n}{2} t} (1+t)^{1-\text{sgn} M} \left\{ e^{\frac{t}{2}} \|\psi_0\|_{H_{(s)}(\mathbb{R}^n)} + \|\psi_1\|_{H_{(s)}(\mathbb{R}^n)} \right\} \\ &\leq C_M e^{(\gamma_0 - \frac{n-1}{2})t} (1+t)^{1-\text{sgn} M} \left\{ \|\psi_0\|_{H_{(s)}(\mathbb{R}^n)} + \|\psi_1\|_{H_{(s)}(\mathbb{R}^n)} \right\}. \end{aligned}$$

We choose  $0 \leq \gamma_0 < \frac{n-1}{2}$  if  $m = n/2$  and  $0 \leq \gamma_0 \leq \frac{n-1}{2}$  if  $m > n/2$ . Thus,  $\psi_0 \in X(R, s, \gamma_0)$ . Theorem 5.1 implies existence of a solution  $\psi \in X(R, s, \gamma)$  with  $\gamma \leq \min \left\{ \gamma_0, \frac{n}{2(\alpha+1)} \right\}$  of the integral equation (5.3) provided that  $R$  is sufficiently small. Theorem 0.1 is proved.  $\square$

## 5.3 Proof of Theorem 0.2

First we consider the case of small  $m < n/2$ . According to Corollary 3.3 the solution  $\psi_0$  of the linear problem (3.2) with “plus” satisfies the following estimate

$$\begin{aligned} \|\psi_0(x, t)\|_{H_{(s)}(\mathbb{R}^n)} &\leq C_M e^{-(\frac{n}{2}-M)t} \int_0^t e^{(\frac{n}{2}-M)b} \|f(x, b)\|_{H_{(s)}(\mathbb{R}^n)} db \\ &\leq C_M e^{-(\frac{n}{2}-M)t} \int_0^t e^{(\frac{n}{2}-M)b} e^{-\gamma_{rhs} b} e^{\gamma_{rhs} b} \|f(x, b)\|_{H_{(s)}(\mathbb{R}^n)} db \\ &\leq C_M \left( \sup_{\tau \in [0, t]} e^{\gamma_{rhs} \tau} \|f(x, \tau)\|_{H_{(s)}(\mathbb{R}^n)} \right) e^{-(\frac{n}{2}-M)t} \int_0^t e^{(\frac{n}{2}-M-\gamma_{rhs})b} db. \end{aligned}$$

Consider three cases:  $\gamma_{rhs} < n/2 - M$ ,  $\gamma_{rhs} = n/2 - M$ , and  $\gamma_{rhs} > n/2 - M$ . In the first case of  $\gamma_{rhs} < n/2 - M$  we have

$$\|\psi_0(x, t)\|_{H_{(s)}(\mathbb{R}^n)} \leq C_M \left( \sup_{\tau \in [0, \infty)} e^{\gamma_{rhs} \tau} \|f(x, \tau)\|_{H_{(s)}(\mathbb{R}^n)} \right) e^{-\gamma_{rhs} t}.$$

If  $\gamma_{rhs} = n/2 - M$ , then

$$e^{(\frac{n}{2}-M)t} \|\psi_0(x, t)\|_{H_{(s)}(\mathbb{R}^n)} \leq C_M t \left( \sup_{\tau \in [0, \infty)} e^{\gamma_{rhs}\tau} \|f(x, \tau)\|_{H_{(s)}(\mathbb{R}^n)} \right).$$

Thus, according to Theorem 5.1 for sufficiently small  $\varepsilon$ , and

$$\sup_{\tau \in [0, \infty)} e^{\gamma_{rhs}\tau} \|f(x, \tau)\|_{H_{(s)}(\mathbb{R}^n)} \leq \varepsilon$$

the problem (0.4), (0.5) has a global solution and

$$\sup_{\tau \in [0, \infty)} e^{\gamma\tau} \|\psi(x, \tau)\|_{H_{(s)}(\mathbb{R}^n)} \leq 2\varepsilon,$$

where  $\gamma < \gamma_{rhs}/(\alpha + 1)$ .

If  $\gamma_{rhs} > n/2 - M$ , then

$$e^{(\frac{n}{2}-M)t} \|\psi_0(x, t)\|_{H_{(s)}(\mathbb{R}^n)} \leq C_M t \left( \sup_{\tau \in [0, \infty)} e^{\gamma_{rhs}\tau} \|f(x, \tau)\|_{H_{(s)}(\mathbb{R}^n)} \right),$$

and

$$\sup_{\tau \in [0, \infty)} e^{\gamma\tau} \|\psi(x, \tau)\|_{H_{(s)}(\mathbb{R}^n)} \leq 2\varepsilon,$$

where  $\gamma < (\frac{n}{2} - M)/(\alpha + 1)$ .

For the large mass  $m \geq n/2$ , due to (2.13), we have

$$\begin{aligned} & \|\psi_0(x, t)\|_{H_{(s)}(\mathbb{R}^n)} \\ & \leq C_M e^{-\frac{n}{2}t} \int_0^t e^{\frac{n}{2}b} \|f(x, b)\|_{H_{(s)}(\mathbb{R}^n)} (1+t-b)^{1-\text{sgn}M} db \\ & \leq C_M \left( \sup_{\tau \in [0, \infty)} e^{\gamma_{rhs}\tau} \|f(x, \tau)\|_{H_{(s)}(\mathbb{R}^n)} \right) e^{-\frac{n}{2}t} \int_0^t e^{(\frac{n}{2}-\gamma_{rhs})b} (1+t-b)^{1-\text{sgn}M} db. \end{aligned}$$

First let  $m = n/2$ , then

$$\|\psi_0(x, t)\|_{H_{(s)}(\mathbb{R}^n)} \leq C_M \left( \sup_{\tau \in [0, \infty)} e^{\gamma_{rhs}\tau} \|f(x, \tau)\|_{H_{(s)}(\mathbb{R}^n)} \right) e^{-\frac{n}{2}t} \int_0^t e^{(\frac{n}{2}-\gamma_{rhs})b} (1+t-b) db.$$

If, in addition,  $\frac{n}{2} > \gamma_{rhs}$ , then

$$\|\psi_0(x, t)\|_{H_{(s)}(\mathbb{R}^n)} \leq C_M \left( \sup_{\tau \in [0, \infty)} e^{\gamma_{rhs}\tau} \|f(x, \tau)\|_{H_{(s)}(\mathbb{R}^n)} \right) e^{-\gamma_{rhs}t}$$

and we choose  $\gamma_0 = \gamma_{rhs}$  and apply Theorem 5.1 which implies existence of a global solution  $\psi \in X(R, s, \gamma)$ , where  $\gamma \leq \min\{\gamma_{rhs}, \frac{n}{2(\alpha+1)}\}$ .

If additionally  $\frac{n}{2} = \gamma_{rhs}$ , then

$$\|\psi_0(x, t)\|_{H_{(s)}(\mathbb{R}^n)} \leq C_M \left( \sup_{\tau \in [0, \infty)} e^{\gamma_{rhs}\tau} \|f(x, \tau)\|_{H_{(s)}(\mathbb{R}^n)} \right) e^{-\gamma_{rhs}t} (1+t)^2.$$

We choose  $\gamma_0 < \gamma_{rhs}$  and apply Theorem 5.1 which implies existence of a global solution  $\psi \in X(R, s, \gamma)$ , where  $\gamma \leq \min\{\gamma_0, \frac{n}{2(\alpha+1)}\}$ .

If additionally  $\frac{n}{2} < \gamma_{rhs}$ , then

$$\begin{aligned} \|\psi_0(x, t)\|_{H_{(s)}(\mathbb{R}^n)} &\leq C_M \left( \sup_{\tau \in [0, \infty)} e^{\gamma_{rhs}\tau} \|f(x, \tau)\|_{H_{(s)}(\mathbb{R}^n)} \right) e^{-\frac{n}{2}t} \int_0^t e^{(\frac{n}{2}-\gamma_{rhs})b} (1+t-b) db \\ &\leq C_M \left( \sup_{\tau \in [0, \infty)} e^{\gamma_{rhs}\tau} \|f(x, \tau)\|_{H_{(s)}(\mathbb{R}^n)} \right) e^{-\frac{n}{2}t} (1+t) \end{aligned}$$

and we choose  $\gamma_0 < n/2$  and  $\gamma \leq \frac{n}{2(\alpha+1)}$ .

Next we consider the case of  $m > n/2$ , then

$$\|\psi_0(x, t)\|_{H_{(s)}(\mathbb{R}^n)} \leq C_M \left( \sup_{\tau \in [0, \infty)} e^{\gamma_{rhs}\tau} \|f(x, \tau)\|_{H_{(s)}(\mathbb{R}^n)} \right) e^{-\frac{n}{2}t} \int_0^t e^{(\frac{n}{2}-\gamma_{rhs})b} db.$$

If additionally  $\frac{n}{2} > \gamma_{rhs}$ , then

$$\|\psi_0(x, t)\|_{H_{(s)}(\mathbb{R}^n)} \leq C_M \left( \sup_{\tau \in [0, \infty)} e^{\gamma_{rhs}\tau} \|f(x, \tau)\|_{H_{(s)}(\mathbb{R}^n)} \right) e^{-\gamma_{rhs}t}$$

and we choose  $\gamma_0 = \gamma_{rhs}$  and  $\gamma \leq \min\{\gamma_{rhs}, \frac{n}{2(\alpha+1)}\}$ .

If additionally  $\frac{n}{2} = \gamma_{rhs}$ , then

$$\|\psi_0(x, t)\|_{H_{(s)}(\mathbb{R}^n)} \leq C_M \left( \sup_{\tau \in [0, \infty)} e^{\gamma_{rhs}\tau} \|f(x, \tau)\|_{H_{(s)}(\mathbb{R}^n)} \right) e^{-\frac{n}{2}t}$$

and we choose  $\gamma_0 < n/2$  and  $\gamma \leq \min\{\frac{n}{2}, \frac{n}{2(\alpha+1)}\} = \frac{n}{2(\alpha+1)}$ .

If additionally  $\frac{n}{2} < \gamma_{rhs}$ , then

$$\|\psi_0(x, t)\|_{H_{(s)}(\mathbb{R}^n)} \leq C_M \left( \sup_{\tau \in [0, \infty)} e^{\gamma_{rhs}\tau} \|f(x, \tau)\|_{H_{(s)}(\mathbb{R}^n)} \right) e^{-\frac{n}{2}t}$$

and we choose  $\gamma_0 = n/2$  and  $\gamma \leq \min\{\frac{n}{2}, \frac{n}{2(\alpha+1)}\} = \frac{n}{2(\alpha+1)}$ . Theorem 0.2 is proved.  $\square$

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